Chapter 24: Systems of differential equations

A linear system of autonomous differential equations can be expressed as:

$\dot{y}_1 = a_{11}y_1 + a_{12}y_2 + b_1$

$\dot{y}_2 = a_{21}y_1 + a_{22}y_2 + b_2$

The system can be solved either through using the substitution method or the direct method. The direct method is more generally used since it can also be used to solve systems with more than two equations.

With I we refer to the **identity matrix:**

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

A^{-1} refers to the **inverse matrix.**

Solving a system of linear, autonomous, first-order differential equations

System form is : $\dot{y}_1 = a_{11}y_1 + a_{12}y_2 + b_1$

$\dot{y}_2 = a_{21}y_1 + a_{22}y_2 + b_2$

1. Write the system in general matrix form

$$\begin{bmatrix}\dot{y}_1(t)\\ \dot{y}_2(t)\end{bmatrix} = \begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}y_1(t)\\ y_2(t)\end{bmatrix} + \begin{bmatrix}b_1\\ b_2\end{bmatrix}$$

   Can also be written as $\dot{y}(t) = Ay(t) + b$

   A is called the **matrix coefficient** and b is called the **vector of terms.**

2. Compute the homogeneous solution:

   - compute the roots (λ) using the equation $\lambda^2 - trA\lambda + \text{Determinant } A = 0$ where $trA = a_{11} + a_{22}$

     The smallest root is used as the first root (λ_1)

   - compute the corresponding **eigenvectors**

     $(A - \lambda_1 I)\, v_1 = 0$ and $(A - \lambda_2 I)\, v_2 = 0$

   - report homogeneous solution

     $y_h(t) =$ $C_1v_1e^{\lambda 1t} + C_2v_2e^{\lambda 2t}$ if λ_1 is not equal to λ_2

     $(C_1 + C_2t)v_1e^{\lambda t} + C_2v_2e^{\lambda t}$ if $\lambda_1 = \lambda_2 = \lambda$

3. Compute the particular solution, which in a system is $\bar{y}$:

   $\bar{y} = -A^{-1}\, b$

4. Report general solution

   $y(t) = y_h(t) + \bar{y}$

5. Compute constants C_1 and C_2 such that the initial conditions are satisfied

6. Report solution

Stability analysis:

- if λ_1 and λ_2 have opposite signs → saddle-point
- if λ_1, λ_2 = - → stable node
- if λ_1, λ_2 = + → unstable node
- if $\lambda_1 = \lambda_2 = \lambda < 0$ → improper stable node
- if $\lambda_1 = \lambda_2 = \lambda > 0$ → improper unstable node
- if λ_1, λ_2 are complex with $a_{11} + a_{22} < 0$ → stable focus
- if λ_1, λ_2 are complex with $a_{11} + a_{22} > 0$ → unstable focus
- if λ_1, λ_2 are complex with $a_{11} + a_{22} = 0$ → center

The steady-state solution to a system of equations is stable only if the characteristic roots are negative. If one of the characteristic roots is positive while the other is negative, then the steady-state equilibrium is unstable and called a **saddle-point equilibrium**.

However, y_1 and y_2 do converge toward the steady-state solutions if the initial conditions satisfy the following equation:

$$y_2 = \frac{r_1 - a_{11}}{a_{12}}(y_1 - \bar{y}_1) + \bar{y}_2$$

In this equation r_1 is the negative root. The points defined by this formula are called the **saddle path.**

A graph of a system with y_1 and y_2 on the axes is called a **phase plane**.

Deriving a phase plane with y_1 and y_2 on the axes:

1. Draw a plane with y_1 on the horizontal axis and y_2 on the vertical axis.

2. Compute isoclines by setting $\dot{y}_1 = 0$ and $\dot{y}_2 = 0$. Draw these lines in the plane

3. Draw the saddle path using the equation on the previous page.

4. Draw arrows of motion with arrows in the graph

   - if $d\dot{y}_i / dy_i < 0$ then y_i moves towards the isocline

   - if $d\dot{y}_i / dy_i > 0$ then y_i moves away from the isocline

The path followed by the pair y_1 and y_2 in the phase plane is called the **trajectory.**