Chapter 10 Rate of Interest and Values

Interest

Interest rates are usually quotes as annual rates, also called nominal rates, even if the actual interest period is different. The interest period is the time between two successive dates when interest is added to the account. To get the periodic rate of interest you need to divide the nominal rate of interest by the number of periods. Therefore the principle increases as follows:

$$S(t) = S_0(1+r)^t$$

To be able to compare interest rates, usually the concept of the *effective yearly rate* and its independent of the value of the initial principle S:

$$R = \left(1 + \frac{r}{n}\right)^n - 1$$

Continuous Compounding

When we assume continuous compounding, it implies that we expect interest to be compounded regularly and frequently. To show how much a principle S will have increased after t years with annual interest r, we use the formula:

$$S(t) = S_0 e^{rt}$$

When calculating the present value of a payment due in the future we have to consider the interest rate. If the interest rate is p& per year and $r = \frac{p}{100}$, an amount K that is payable in t years has the present value of:

- $K(1+r)^{-t}$, in the case of annual interest payments, and
- Ke^{-rt}, in the case of continuous compounding

Geometric series

In finance and economics many applications of geometric series exist. There are both finite and infinite series.

$k-1$

- Summation for infinite geometric series: $a + ak + ak^2 + \cdots + ak^{n-1} + \cdots = \frac{a}{1-k}$ or using other notation $\sum_{n=1}^{\infty} ak^{n-1} = \frac{a}{1-k}$, in both cases only when $|k| < 1$

An annuity is a sequence of equal payments made at fixed periods of time over some period. To calculate the present value of an annuity we use the formula for a finite geometric series:

$$P_n = \frac{a}{1+r} + \cdots + \frac{a}{(1+r)^n} = \frac{a}{r}\left[1 - \frac{1}{(1+r)^n}\right]$$

The future value of an annuity is the accumulated value in the account after n periods:

$$F_n = \frac{a}{r}[(1+r)^n - 1]$$

The two formulas for the present and future value are based on the idea of discrete accumulation. In the case of a continuous income stream, we have to use integrals to calculate the present and future values:

$$Present\ Discounted\ Value = \int_0^T f(t)e^{-rt}\,dt$$

$$Future\ Discounted\ Value = \int_0^T f(t)e^{-r(T-t)}\,dt$$

$-r(t-s)dt$, *over the interval* $[s,T]$

Repayment of Mortgage

Usually mortgages are paid back at a fixed interest rate, with equal payments due each period. The payments continue until the loan is paid off. The fixed payments go partly to the interest and partly to the outstanding principle. In the beginning the interest rate is large, but small in the last periods.

An example can be illustrative: A person has taken a mortgage for € A that he will pay over t installments at the rate of r% compounded annually. How much will he pay per installment? Let p be the amount for each installment and using the formula for a finite geometric serie:

$$\frac{p}{r} \times 100 \left[1 - \frac{1}{(1+\frac{r}{100})^n}\right] = A$$

Let us use the example of someone who has a mortgage of € 100000 that he will pay over 4 installments at the rate of 20% compounded annually. How much will he pay per installment?

$$\frac{p}{20} \times 100 \left[1 - \frac{1}{\left(1+\frac{20}{100}\right)^4}\right] = 100000$$

$p \times 5\left[1 - \frac{1}{(1.20)^4}\right] = 100000$, $p = \frac{100000}{5} \times 0.5177$, $p = 10354$

Now to find the amount per installment or p we can just use:

$$p = \frac{rA}{1-(1+r)^{-n}}$$

To find the number of periods required to pay back the loan at given amount per installment we can use:

$$n = \frac{lnp - \ln(p - rA)}{\ln(1+r)}$$

Internal Rate of Return

The internal rate of return is defined as an interest rate that makes the present value of all payments equal to zero. Just remember the following formula in which A is the initial investment and the returns per period are $p_1, p_2 \dots\dots p_n$ for n periods, The rate of return is r. Then:

$$A = \frac{p_1}{(1+r)^1} + \frac{p_2}{(1+r)^2} + \cdots \dots + \frac{p_n}{(1+r)^n}$$

To make the calculation easier assume that $(1+r)^{-1} = x$ and then rewrite the formula $A = p_1 x + p_2 x^2 + \cdots \dots + p_n x^n$. Then you can first solve for x and substitute $(1+r)^{-1} = x$ to find the value of r.

Difference equations

Difference equations are equations that relate certain quantities at different discrete moments of time. The initial period is t=0. A very simple difference equation is:

$x_{t+1} = a x_t$