Chapter 11 Many Variables

Two Variables

A function f of two real variables x and y with domain D is a rule that assigns a specified number $f(x,y)$ to each point (x,y) in D. In such a function there are two independent variables, and one dependent variable. The range still corresponds to the dependent variable.

The most widely used example in economics is the Cobb-Douglas function:
$F(x,y) = Ax^a y^b$

When differentiating a function with more than one variable, we first have to take so-called *partial derivatives* with respect to each independent variable. So if $z = f(x,y)$, then $\frac{\partial z}{\partial x}$ is the derivative of the function with respect to x, while y is kept constant. And $\frac{\partial z}{\partial y}$ is the derivative of the function with respect to y, while x is kept constant.

The formal definitions of the partial derivatives are:

- $f_1'(x,y) = \lim_{h \to 0} \frac{f(x+h,y)-f(x,y)}{h}$, this partial derivative is approximately equal to the change in the original function $f(x,y)$ per unit increase in x, holding y constant.
- $f_2'(x,y) = \lim_{k \to 0} \frac{f(x,y+k)-f(x,y)}{k}$, this partial derivative is approximately equal to the change in the original function $f(x,y)$ per unit increase in y, holding x constant.

The derivatives above are the first-order derivatives. We can also find derivatives of a higher order. The second-order derivatives are denotes as follows:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}, \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$
$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

We can see that on the second line the partial derivatives are actually the same. That means that it does not matter to which variable you differentiate first, if you differentiate for the other after. The order is not relevant.

If me move from a function with one independent variable to a function with two independent variables, then the geometric representation transforms from two-dimensional to three-dimensional. Instead of two axes, now there will be three. In the case of more variables we do not have the means to represent them as a graph visually.

Surfaces and Distance

In a three-dimensional figure a function represents a surface instead of a line. For a function with three variables (x,y,z), the general equation for a plane in space is given by $px + qy + rz = s$, in which $p, q, r \text{ and } s$ are constants.

We can interpret each independent variable and each axis to represent a product, and the constants the corresponding prices. Then the surface is called budget plane, where p, q and r are cost of goods/unit and s is the total value. For a three-dimensional space, we can find the distance between two arbitrary points (p_1, q_1, r_1) and (p_2, q_2, r_2) in that space. The distance is given by:

$$distance = \sqrt{(p_1, -p_2)^2 + (q_1 - q_2)^2 + (r_1 - r_2)^2}$$

The equation for a sphere is given for one center point (u, v, w) and using radius r:

$$(x-u)^2 + (y-v)^2 + (z-w)^2 = r^2$$

More Variables

To discuss functions with more than two independent variables we have to introduce the concept of a *vector*. Any ordered collection of numbers $(x_1, x_2, x_3, \dots, x_n)$ is called an n-vector. Vectors are usually denoted by bold letters. Hence, we can simply write $\boldsymbol{x} = (x_1, x_2, x_3, \dots, x_n)$. Formally, given any set D of n-vectors, a function of f of n variables $x_1, \dots, x_n$ with domain D is a rule that assigns a specifiek number $f(\boldsymbol{x}) = f(x_1, \dots, x_n)$ to each vector $\boldsymbol{x} = (x_1, x_2, x_3, \dots, x_n)$.

Functions with multiple variables can be linear, quadratic or exponential. Do not forget that an exponential function such as the Cobb-Douglas function can be transformed into a log-linear function by taking the natural logarithms:

$$F(x,y) = Ax^a y^b \rightarrow \ln F = \ln A + a \ln x + b \ln y$$

Again any function of n variables that can be constructed from continuous functions by combining addition, subtraction, multiplication, division and functional composition is continuous wherever it is defined.

Partial Derivatives

We can extend the differentiation process to functions with multiple variables. A way of representing the resulting sets of partial derivatives to the second-degree is in the *Hessian Matrix:*

$$f''(\boldsymbol{x}) = \begin{pmatrix} f''_{11}(\boldsymbol{x}) & \cdots & f''_{1n}(\boldsymbol{x}) \\ \vdots & \ddots & \vdots \\ f''_{n1}(\boldsymbol{x}) & \cdots & f''_{nn}(\boldsymbol{x}) \end{pmatrix}$$

Young's theorem says that if all the mth-order partial derivatives of a function are continuous, and if any two of them involve differentiating with respect to each of the variables the same number of times, then they are necessarily equal:

$$\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right) = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right)$$

Partial elasticities

Also in the case of functions with multiple variables we can find elasticities. If $z = f(x, y)$ then we define the partial elasticity of z with respect to x and y by:

$$El_x z = \frac{x}{z}\frac{\partial z}{\partial x} = \frac{\partial \ln z}{\partial \ln x}, and\ El_y z = \frac{y}{z}\frac{\partial z}{\partial y} = \frac{\partial \ln z}{\partial \ln y}$$

These expressions can be generalized to functions with more than two variables.