Chapter 12 Comparative Statistics

Chain Rule

When $z = F(x, y)$ with $x = f(t)$ and $y = g(t)$, then we can use the simple chain rule to find the *total derivative*:

$$\frac{dz}{dt} = F_1'(x,y)\frac{dx}{dt} + F_2'(x,y)\frac{dy}{dt}$$

The chain rule can also be applied to problems with multiple variables. If $z = F(x, y)$ and $x = f(t, s)$ and $y = g(t, s)$, then the composite function is $z{=}F\ f\ t{,}s{,}g\ t{,}s$, then the two partial derivatives are :

$$\frac{dz}{dt} = F_1'(x,y)\frac{dx}{dt} + F_2'(x,y)\frac{dy}{dt}$$
$$\frac{dz}{ds} = F_1'(x,y)\frac{dx}{ds} + F_2'(x,y)\frac{dy}{ds}$$

We can transform these expressions into the general case with n variables:

$$\frac{dz}{dt_j} = \frac{\partial z}{\partial x_1}\frac{dx_1}{dt_j} + \frac{\partial z}{\partial x_2}\frac{dx_2}{dt_j} + \cdots + \frac{\partial z}{\partial x_n}\frac{dx_n}{dt_j}$$

Implicit Differentiation

Economists often have to differentiate functions that are defined implicitly by an equation, for example: $F\ x, y{=}c$ and y is defined by an equation. To find the slope of the function $y = f(x)$ we can use:

$$y' = -\frac{F_1'(x,y)}{F_2'(x,y)} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$$

The process to find the second derivative of this derivative seems cumbersome, but is in essence the simple application of the quotient rule:

1. Write the expression of y' above as $y' = -\frac{G(x)}{H(x)}$.
2. Now differentiate to x: $y'' = -\frac{G'(x)H(x) - G(x)H'(x)}{[H(x)]^2}$
3. Both $G(x)$ and $H(x)$ are composite functions and therefore we can find the derivatives as follows:
    a. $G'(x) = F_{11}''(x,y) \cdot 1 + F_{12}''(x,y) \cdot y'$
    b. $H'(x) = F_{21}''(x,y) \cdot 1 + F_{22}''(x,y) \cdot y'$
4. Remember that $F_{12}'' = F_{21}''$, and replace y' by $-\frac{F_1'}{F_2'}$, and insert all the results into y'' that we have found in step 2.
5. $y'' = -\frac{1}{(F_2')^3}[F_{11}''(F_2')^2 - 2F_1''F_1'F_2' + F_{22}''(F_1')^2]$ is the result we find

General Cases

In general, what we know about the partial derivatives when $z = f(x, y)$ and c is a constant is the following:

$$F(x,y,z) = c \implies z'_x = -\frac{F'_x}{F'_z}, \quad z'_y = -\frac{F'_y}{F'_z} \text{ for } F'_z \neq 0$$

For the case with n variables we find:
∂z

Elasticity of Substitution

Even though in economics the slope of a curve is often downwards sloping, we change the sign of the slope and call it the *marginal rate of substitution of y for x (MRS):*

$$R_{xy} = \frac{F'_x(x,y)}{F'_y(x,y)} = -y' \approx -\frac{\Delta y}{\Delta x}$$

The elasticity of substitution σ_{yx} is the elasticity of the fraction $\frac{y}{x}$ with respect to the MRS. It is therefore more or less the percentage change in this fraction when we move along the level curve far enough for the MRS to change with 1%.

$$\sigma_{yx} = \mathrm{El}_{R_{xy}}\left(\frac{y}{x}\right)$$

Homogeneous Functions

A function $f(x,y)$ is said to be homogeneous of degree k if, for all (x, y) in D, the following holds for all $t > 0$:
$$f(tx,ty) = t^k f(x,y)$$

In words, multiplying both variables x and y by a factor t, will multiply the entire function by the factor t^k. One of the important properties of homogeneous functions is defined by *Euler's theorem*, which says that if a function is homogeneous of degree k, then:
$$xf_1'(x,y) + yf_2'(x,y) = kf(x,y)$$

In a more general formulation Euler's theorem looks like this:

$$\sum_{i=1}^{n} x_i f_i'(\boldsymbol{x}) = kf(\boldsymbol{x})$$

Given that f is a differentiable function in an open domain D.

A function is homothetic provided that:

$$\boldsymbol{xy} \in K, \; f(\boldsymbol{x}) = f(\boldsymbol{y}), \; t > 0 \implies f(t\boldsymbol{x}) = f(t\boldsymbol{y})$$

Every homogeneous function of degree k is also a homothetic function. More generally we can say that if H is a strictly increasing function and f is homogeneous of degree k, then $F\ \boldsymbol{x}=H\ f\ \boldsymbol{x}$.

Linear Approximation

The Linear approximation to $f(x,y)$ about (x_0,y_0) is given by the formula:

$$f(x,y) \approx f(x_o,y_o) + f'_1(x_o,y_o)(x-x_o) + f'_2(x_o,y_o)(y-y_o)$$

Where, $f'_1(x_o,y_o)$, stands for first derivative of (x_o,y_o) in terms of x and $f'_2(x_o,y_o)$ stands for the first derivative of (x_o,y_o), in terms of y.

We can generalize this equation for functions of several variables. For a function $z = f(\boldsymbol{x}) = f(x_1, \dots\dots x_n)\ about\ \ \boldsymbol{x}^0 = (x_1^0, \dots\dots, x_n^0)$, the linear approximation is:

$$f(\boldsymbol{x}) \approx f(\boldsymbol{x}^0) + f'_1(\boldsymbol{x}^0)(x - x^0) + \dots\dots\dots. + f'_n(\boldsymbol{x}^0)(x_n - x_n^0)$$

In a situation with two independent variables we are no longer considering a two-dimensional graph, but a three-dimensional space. Therefore we can no longer speak about a tangent line, but have to consider a tangent plane. The tangent plane has the equation:

$$z - c = f'_1(a,b)(x-a) + f'_2(a,b)(y-b)$$

Differentials

Consider a function $z = f(x,y)$. When x and y change a bit, the resulting change in z is called the increment:

$\Delta z = f(x+dx, y+dy) - f(x,y)$

The differential of the function is denoted by dz or df and is:

$'\ x,ydy$

If dx and dy are small in absolute values then, $dz \approx \Delta z$. The following (familiar) rules apply to differentials (f and g are functions):

- $d\ af+bg=a\ df+b\ dg$
- $d(fg) = g\ df + f\ dg$
- $d\left(\frac{f}{g}\right) = \frac{g\ df - f\ dg}{g^2}$
- $z = g(f(x,y)) \Rightarrow dz = g'(f(x,y))df$

Systems of Equations

In many economic models a large number of variables relate to each other through a system of simultaneous equations. To find the number of degrees of freedom we use the counting rule. First count the number of variables or n, and then how many number of 'independent' equations (i) there are. If $n > i$ then $n - i$ is the number of degrees of freedom. If $n < i$ then there is no solution to that system.

A system of equations with n number of variables will have k degrees of freedom. k is the number of variables that can be chosen freely. And $n - k$ is the number of variables whose value can be found when the value of k is decided upon.

In general, a system with as many equations as variables is usually consistent, in other words, has solutions, but it may have several solutions. Usually there is not a unique solution unless there are exactly as many equations as unknowns.

Basics of Differentiating Systems of Equation

To differentiate a system of equations, the first step to take is to differentiate both sides of the equation with respect to their variables. Consider for example the following two equations: $2a + 2b = 3x - 2y$. Step one would result in $2da + 2db = 3dx - 2dy$.
The next step is to find the values of $da \text{ and } db$ in terms of $dx \text{ and } dy$ by using the two equations. Then we have to find $a'_x \text{ and } b'_x \text{ also } a'_y \text{ and } b'_{y}$. Just remember that if the equation for da looks like $da = 3dx + 4dy$ then the result would be simply $a'_x = 3 \text{ and } a'_y = 4$. The last thing to do is to substitute the points that are given to find the exact values