Chapter 14 Constrained Optimization

Lagrangian Multiplier Method

A maximization problem can be constrained. An example from economics is the maximization of a utility function under the constraint of a budget. We would write that problem as $\max u(x,y)$ subject to $px + y = m$. Economists make use of the Lagrange multiplier method to solve such complicated constrained maximization problems:

To find the solutions of the problem maximize, or minimize, $f(x,y)$ subject to $g(x,y) = c$ proceed as follows:

1. Write down the Lagrangian: $\mathcal{L}(x,y) = f(x,y) - \lambda(g(x,y) - c)$ where λ is a constant.
2. Differentiate this Lagrangian $\mathcal{L}$ with respect to x and y and equate both partial derivatives to zero.
3. These two equations together with the constraint give this system of equations (FOCs):
    a. $\mathcal{L}_1'(x,y) = f_1'(x,y) - \lambda g_1'(x,y) = 0$
    b. $\mathcal{L}_2'(x,y) = f_2'(x,y) - \lambda g_2'(x,y) = 0$
    c. $g(x,y) = 0$
4. Solve these three equations simultaneously for the three unknowns x, y and λ. These triplets (x, y, λ) are solution candidates.

The Lagrange multiplier λ is the rate at which the optimal value of the objective function changes with respect to changes in the constraint constant. Economists call λ the shadow price.

This method is based on Lagrange's theorem. For completeness, we formally describe the theorem:

Suppose that $f(x,y)$ and $g(x,y)$ have continuous partial derivatives in a domain A of the xy-plane, and that the point (x_0,y_0) is both an interior point of A and a local extreme point for $f(x,y)$ subject to the constraint $g(x,y) = c$. Suppose further that $g_1'(x_0,y_0)$ and $g_2'(x_0,y_0)$ are not both 0. Then there exists a unique number λ such that the Lagrangian $\mathcal{L}(x,y) = f(x,y) - \lambda(g(x,y) - c)$ has a stationary point at (x_0,y_0).

Sufficient conditions

The process above provides us with the necessary conditions for the solution. In order to confirm the found candidates as solutions, we need to carefully check the nature of the points.

We know that if the Lagrangian is concave, then the point (x_0, y_0) is a maximum. If the Lagrangian is convex, then the point (x_0, y_0) is a minimum. Formally, we need to make the following calculation. Point (x_0, y_0) is a:

- local max if $(f''_{11} - \lambda g''_{11})(g'_2)^2 - 2(f''_{12} - \lambda g''_{12})g'_1 g'_2 + (f''_{22} - \lambda g''_{22})(g'_1)^2 < 0$
- local min if $(f''_{11} - \lambda g''_{11})(g'_2)^2 - 2(f''_{12} - \lambda g''_{12})g'_1 g'_2 + (f''_{22} - \lambda g''_{22})(g'_1)^2 > 0$

It is possible and straightforward to complicate the Lagrangian by adding constraints or variables.

Nonlinear Programming

Problems of nonlinear programming concern inequalities. A simply inequality constraint is when certain variables cannot be negative, for example $x_i \geq 0$. The nonlinear programming problem is denoted as:

$$\max f(x, y) \text{ subject to } g(x, y) \leq c$$

The Lagrangian method does not change, but the candidates that are to be found, should satisfy the inequality constraint.

The problem can also involve a set of inequality constraints:

$$\max f(x_i, \ldots, x_n) \text{ subject to } \begin{cases} g_1(x_i, \ldots, x_n) \leq c_1 \\ \ldots \\ g_m(x_i, \ldots, x_n) \leq c_m \end{cases}$$

The set of vectors $\boldsymbol{x} = (x_i, \ldots, x_n)$ that satisfy all the constraints is called the *admissible set* or the *feasible set*.