Chapter 15 Matrices and Vectors

Matrices

A system of equations is consistent if it has at least one solution. When the system has no solutions at all, then it is called inconsistent.

A matrix is a rectangular array of numbers considered as a mathematical object. Matrices are often used to solve systems of equations. When the matrix consists of m rows and n columns then the matrix is said to have the order $m \times n$. All the numbers in a matrix are called elements or entries. If $m = n$ then the matrix is called a square matrix. The main diagonal runs from the top left to the bottom right and are the elements $a_{11}, a_{22}, a_{33} \dots$

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{m1} & \dots & a_{mn} \end{pmatrix}$$

A matrix with only one row, or only one column, is called a *vector*. And we can distinguish between a row vector and a column vector. A vector is usually denoted by a bold letter $\boldsymbol{x}$.

One can transform a system of equations into a matrix or order the coefficients of system in a matrix.

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$
$$\dots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

This system can be written, now in short form, as: $\boldsymbol{Ax} = \boldsymbol{b}$.

Matrix Operations

Two matrices $\boldsymbol{A} = (a_{ij})_{m\times n}$ and $\boldsymbol{B} = (b_{ij})_{m\times n}$ are said to be equal if all $a_{ij} = b_{ij}$, or in words, if they have the same order and if all corresponding entries are equal. Otherwise they are not equal and we write $\boldsymbol{A} \neq \boldsymbol{B}$.

The reasoning for addition and multiplication by a constant is straightforward.

$$\boldsymbol{A} + \boldsymbol{B} = (a_{ij})_{m\times n} + (b_{ij})_{m\times n} = (a_{ij} + b_{ij})_{m\times n}$$

$$\alpha\boldsymbol{A} = \alpha(a_{ij})_{m\times n} = (\alpha a_{ij})_{m\times n}$$

The rules that are related to these two operations are:

- $(\boldsymbol{A}+\boldsymbol{B})+\boldsymbol{C}=\boldsymbol{A}+(\boldsymbol{B}+\boldsymbol{C})$
- $\boldsymbol{A}+\boldsymbol{B}=\boldsymbol{B}+\boldsymbol{A}$
- $\boldsymbol{A}+\boldsymbol{0}=\boldsymbol{A}$
- $\boldsymbol{A}+(-\boldsymbol{A})=\boldsymbol{0}$
- $(\alpha+\beta)\boldsymbol{A}=\alpha\boldsymbol{A}+\beta\boldsymbol{B}$
- $\alpha(\boldsymbol{A}+\boldsymbol{B})=\alpha\boldsymbol{A}+\alpha\boldsymbol{B}$

Matrix Multiplication

For multiplication of two matrices, suppose that $\boldsymbol{A}=(a_{ij})_{m\times n}$ and $\boldsymbol{B}=(b_{ij})_{m\times n}$. Then the product $\boldsymbol{C}=\boldsymbol{AB}$ is the $m\times p$ matrix $\boldsymbol{C}=(c_{ij})_{m\times p}$. The element in the i'th row and j'th column is the product of:

$$c_{ij}=\sum_{r=1}^{n} a_{ir}b_{rj}=a_{i1}b_{1j}+a_{i2}b_{2j}+\cdots+a_{ik}b_{kj}+\cdots+a_{in}b_{nj}$$

To help visualizing this summation, have a look at the following multiplication of matrices:

$$\boldsymbol{A}=\begin{pmatrix}0 & 1 & 2\\ 2 & 3 & 1\\ 4 & -1 & 6\end{pmatrix}, \boldsymbol{B}=\begin{pmatrix}3 & 2\\ 1 & 0\\ -1 & 1\end{pmatrix}$$

$$\boldsymbol{AB}=\begin{pmatrix}0 & 1 & 2\\ 2 & 3 & 1\\ 4 & -1 & 6\end{pmatrix}\begin{pmatrix}3 & 2\\ 1 & 0\\ -1 & 1\end{pmatrix}=\begin{pmatrix}-1 & 2\\ 8 & 5\\ 5 & 14\end{pmatrix}$$

In this case $\boldsymbol{AB}$ is defined but $\boldsymbol{BA}$ would not be defined, because in that case the number of elements in the rows of B does not match the number of elements in the columns of A. Even if they are both defined, they are not automatically equal.

There are rules for matrix multiplication:

- $(\boldsymbol{AB})\boldsymbol{C}=\boldsymbol{A}(\boldsymbol{BC})$, Associative law
- $\boldsymbol{A}(\boldsymbol{B}+\boldsymbol{C})=\boldsymbol{AB}+\boldsymbol{AC}$, Left distributive law
- $(\boldsymbol{A}+\boldsymbol{B})\boldsymbol{C}=\boldsymbol{AC}+\boldsymbol{BC}$, Right distributive law
- $(\alpha\boldsymbol{A})\boldsymbol{B}=\boldsymbol{A}(\alpha\boldsymbol{B})=\alpha(\boldsymbol{AB})$
- $\boldsymbol{A}^n=\boldsymbol{AA}\ldots\boldsymbol{A}$, A is repeated n times

There are some dangerous mistakes that are often maid:

- $\boldsymbol{AB}\neq\boldsymbol{BA}$
- $\boldsymbol{AB}=\boldsymbol{0}$ Does not imply that either $\boldsymbol{A}$ or $\boldsymbol{B}$ is $\boldsymbol{0}$
- $\boldsymbol{AB}=\boldsymbol{AC}$ And $\boldsymbol{A}\neq\boldsymbol{0}$ do not imply that $\boldsymbol{B}=\boldsymbol{C}$

The Identity Matrix and the Transpose

The identity matrix of order n, denoted by $\boldsymbol{I}_n$, is the matrix having only ones along the main diagonal and zero's elsewhere:

$$\boldsymbol{I}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The identity matrix is in practice the equivalent of 1 in the numerical system, because $\boldsymbol{A}\boldsymbol{I}_n = \boldsymbol{I}_n\boldsymbol{A} = \boldsymbol{A}$.

The transpose of matrix $\boldsymbol{A}$, $\boldsymbol{A}'$, is the mirror matrix of $\boldsymbol{A}$. More formally, $\boldsymbol{A}'$ is defined as the $n \times m$ matrix whose first column is the first row of $\boldsymbol{A}$, whose second column is the second row of $\boldsymbol{A}$, and so on. Thus:

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \Rightarrow \boldsymbol{A}' = \begin{pmatrix} a_{11} & \cdots & a_{m1} \\ \vdots & \ddots & \vdots \\ a_{1n} & \cdots & a_{mn} \end{pmatrix}$$

The rules for transposition are:

- $(\boldsymbol{A}')' = \boldsymbol{A}$
- $(\boldsymbol{A} + \boldsymbol{B})' = \boldsymbol{A}' + \boldsymbol{B}'$
- $(\alpha\boldsymbol{A})' = \alpha\boldsymbol{A}'$
- $(\boldsymbol{A}\boldsymbol{B})' = \boldsymbol{B}'\boldsymbol{A}'$

A matrix is called symmetric when $\boldsymbol{A} = \boldsymbol{A}'$.

Gaussian Elimination

One method of solving systems of equations is by elimination of the unknowns. Elementary row operations can transform equations in such a way that unknowns can be eliminated. There are three kinds of elementary row operations:

1. Interchange any pair of rows
2. Multiply any row by a scalar
3. Add any multiple of one row to a different row

Strictly the Gaussian method of elimination involves three steps:

1. Make a staircase with 1 as the coefficient for each nonzero leading entry.
2. Produce 0's above each leading entry.
3. Express the unknowns in terms of those unknowns that do not occur as leading entries. The number of unknowns that can be chosen freely is the number of *degrees of freedom*.

See page 565 to 569 for extensive numerical examples.

Vectors

The numbers in a vector are called the components, or coordinates of the vector. A vector is just a specific type of matrix and therefore, the algebra of matrices is also valid for vectors:

- Two vectors are equal only if all their corresponding components are equal.
- The sum of two n-vectors is found by adding each component of the first vector to the corresponding component in the other vector.
- Any vector can be multiplied by a real number.
- The difference between two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is defined as $\boldsymbol{a} - \boldsymbol{b} = \boldsymbol{a} + (-\mathbf{1})\boldsymbol{b}$.

The so-called *inner product* of the n-vectors $\boldsymbol{a} = (a_1, a_2, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, b_2, \ldots, b_n)$ is defined as:

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n = \sum_{i=1}^{n} a_i b_i$$

The rules for the inner product are, if $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are n-vectors and α is a scalar then:

- $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$
- $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$
- $(\alpha\boldsymbol{a}) \cdot \boldsymbol{b} = \boldsymbol{a} \cdot (\alpha\boldsymbol{b}) = \alpha(\boldsymbol{a} \cdot \boldsymbol{b})$
- $\boldsymbol{a} \cdot \boldsymbol{a} > 0 \iff \boldsymbol{a} \neq \mathbf{0}$

For a small section on geometric interpretations of vectors, have a look at page 575.