Chapter 16 Determinants and Inverse Matrices

Determinants

The value of the determinant of a matrix $\boldsymbol{A}$ denoted by $\det(\boldsymbol{A})$ or $|\boldsymbol{A}|$ determines if there is a unique solution to the corresponding system of equations.

$$|\boldsymbol{A}| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

This particular example is said to have the order two. Calculating the determinant of order two is simple. When a determinant is of a higher order though, the calculations become more extensive. Take for example this determinant of order 3:

$$|\boldsymbol{A}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

This expression is actually the same expression as:

$$|\boldsymbol{A}| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

A numerical example illustrates the procedure. Calculate:

$$|\boldsymbol{A}| = \begin{vmatrix} 3 & 0 & 2 \\ -1 & 1 & 0 \\ 5 & 2 & 3 \end{vmatrix}$$

The solution is:

$$|\boldsymbol{A}| = 3\begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} - 0\begin{vmatrix} -1 & 0 \\ 5 & 3 \end{vmatrix} + 2\begin{vmatrix} -1 & 1 \\ 5 & 2 \end{vmatrix} = 3 \cdot 3 - 0 + 2 \cdot (-2-5) = -5$$

To determine the sign of any term in the sum above, mark in the array all the elements appearing in that term. Join all possible pairs of these elements with lines. These lines will then either fall or rise to the right. If the number of rising lines is even, then the corresponding term is assigned a plus sign, if it is odd a minus sign. (Illustration on page 593)

In fact, the definition of the determinant is a sum of $n!$ terms where each term is the product of n elements of the matrix, with one element from each row and one element from each column. Moreover, every product of n factors in which each row and each column is represented exactly once, must appear in the sum.

Basic Rules
Consider the $n \times n$ matrix $\boldsymbol{A}$, then:

- If all the elements in a row or column of $\boldsymbol{A}$ are 0, then $|\boldsymbol{A}| = 0$.
- $|\boldsymbol{A}'| = |\boldsymbol{A}|$ where $\boldsymbol{A}'$ is the transpose of $\boldsymbol{A}$.
- If all the elements in a single row or column of $\boldsymbol{A}$ are multiplied by a certain number, then the determinant is also multiplied by this same number.
- If two rows or two columns of $\boldsymbol{A}$ are interchanged, the determinant changes sign, but the absolute value remains unchanged.
- If two of the rows or columns of $\boldsymbol{A}$ are proportional, then $|\boldsymbol{A}| = \boldsymbol{0}$.
- The value of the determinant of $\boldsymbol{A}$ is unchanged if a multiple of one row or one column is added to a different row or column of $\boldsymbol{A}$.
- The determinant of the product of two $n \times n$ matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ is the product of the determinants of each of the factors: $|\boldsymbol{AB}| = |\boldsymbol{A}| \cdot |\boldsymbol{B}|$
- If α is a real number, then $|\alpha \boldsymbol{A}| = \alpha^n |\boldsymbol{A}|$

Note that in general $|\boldsymbol{A} + \boldsymbol{B}| \neq |\boldsymbol{A}| + |\boldsymbol{B}|$.

Expansion by Cofactors
Each term of the determinant of matrices contains one element from each row and one element from each column. The expansion of $|\boldsymbol{A}|$ in terms of elements of the row i is called the *cofactor expansion*:

$$|\boldsymbol{A}| = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{ij}C_{ij} + \cdots + a_{in}C_{in}$$

the cofactor expansion can also be found for a certain column. To find each cofactor C_{ij} there is a simple procedure to apply to the determinant. First delete row i and column j to arriver at a determinant of order $n-1$, called a *minor*. Then, multiply the minor by the factor $(-1)^{i+j}$:

$$C_{ij} = (-1)^{i+j} \begin{vmatrix} a_{11} & \cdots & a_{1,j-1} & \cancel{a_{1j}} & a_{1,j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2,j-1} & \cancel{a_{2j}} & a_{2,j+1} & \cdots & a_{2n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \cancel{a_{i1}} & & \cancel{a_{i,j-1}} & \cancel{a_{ij}} & \cancel{a_{i,j+1}} & & \cancel{a_{in}} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & \cancel{a_{nj}} & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix}$$

Inverse

Given a matrix $\boldsymbol{A}$ we say that $\boldsymbol{X}$ is an inverse of $\boldsymbol{A}$, and vice versa, if there exists a matrix $\boldsymbol{X}$ such that:

$$\boldsymbol{AX} = \boldsymbol{XA} = \boldsymbol{I}$$

$\boldsymbol{I}$ is the identity matrix. In this case the matrix $\boldsymbol{A}$ is said to be *invertible*. But note that only square matrices have inverses. A square matrix is said to be singular if its determinant equals zero and nonsingular if its determinant does not equal zero.

A matrix has an inverse if and only if it is nonsingular:

$\boldsymbol{A}\ has\ an\ inverse \iff |\boldsymbol{A}| \neq 0$

If a matrix has an inverse, then it is unique. So, assuming that the determinant is nonsingular, the following result has been proven:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \boldsymbol{A}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

The inverse has four properties. Let $\boldsymbol{A}$ and B be invertible $n \times n$ matrices, then:

1. $\boldsymbol{A}^{-1}$ is invertible and $(\boldsymbol{A}^{-1})^{-1} = \boldsymbol{A}$
2. $\boldsymbol{AB}$ is invertible and $(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$
3. The transpose $\boldsymbol{A}'$ is invertible and $(\boldsymbol{A}')^{-1} = (\boldsymbol{A}^{-1})'$
4. $(c\boldsymbol{A})^{-1} = c^{-1}\boldsymbol{A}^{-1}$ whenever c is not zero

Provided that $|\boldsymbol{A}| \neq 0$, the following holds:

$$\boldsymbol{AX} = \boldsymbol{B} \iff \boldsymbol{X} = \boldsymbol{A}^{-1}\boldsymbol{B}$$
$$\boldsymbol{YA} = \boldsymbol{B} \iff \boldsymbol{Y} = \boldsymbol{BA}^{-1}$$

Cramer's Rule

Cramer's rule is that a general linear system of equations with n equations and n unknowns has a unique solution if and only if $\boldsymbol{A}$ is nonsingular, hence if the determinant does not equal zero.

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\ldots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

The solution is:

$$x_1 = \frac{D_1}{|\boldsymbol{A}|}, x_2 = \frac{D_2}{|\boldsymbol{A}|}, \ldots, x_n = \frac{D_n}{|\boldsymbol{A}|}$$

Where D denotes the determinant obtained from the determinant of A by replacing a column by the b_n's and then calculating its value through cofactor expansion.

If all the b equal zero, then the system of equations is called homogeneous. A homogeneous system will always have the so-called *trivial solution* which is $x_1 = x_2 = \cdots = x_n = 0$. Often one is interested in finding nontrivial solutions of a homogeneous system.