Chapter 17 Linear Programming

General

The general linear programming problem is that of maximizing or minimizing the objective function:

$$z = c_1x_1 + \cdots + c_nx_n$$

This objective function is subject to a set of inequality constraints:

$$a_{11}x_1 + \cdots + a_{1n}x_n \leq b_1$$
$$a_{21}x_1 + \cdots + a_{2n}x_n \leq b_2$$
$$\dots\dots\dots$$
$$a_{m1}x_1 + \cdots + a_{mn}x_n \leq b_m$$

Usually it is explicitly assumed that the variables cannot be negative, the nonnegativity constraint. The vector of n solutions, that satisfies these constraints, is called the feasible or admissible vector.

Duality Theory

When an economist is confronted with an optimization problem there is are two ways of approaching the problem. A maximization problem has a mirror that is a minimization problem. For example, if the problem involves the allocation of scarce resources he can try to maximize the production constraint by the available scarce resources, or he can try to minimize the use of resources given a level of production. Hence, there is a duality involved.

In general, consider the general linear programming problem, also called the *primal* problem:

$$\max c_1x_1 + \cdots + c_nx_n \;\; s.t. \;\; \begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n \leq b_1 \\ \dots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \leq b_m \end{cases}$$

Then the *dual* problem is:

$$\min b_1u_1 + \cdots + b_nu_n \;\; s.t. \;\; \begin{cases} a_{11}u_1 + \cdots + a_{1n}u_n \leq c_1 \\ \dots \\ a_{m1}u_1 + \cdots + a_{mn}u_n \leq c_m \end{cases}$$

For both problems the nonnegativity constraint holds as well.

Suppose the primal problem has an optimal solution, then the dual problem also has an optimal solution and the corresponding values of the objective functions are equal. If the prima has no bounded optimum, then the dual has no feasible solution. Symmetrically, if the primal problem has no feasible solution, then the dual has no bounded optimum.