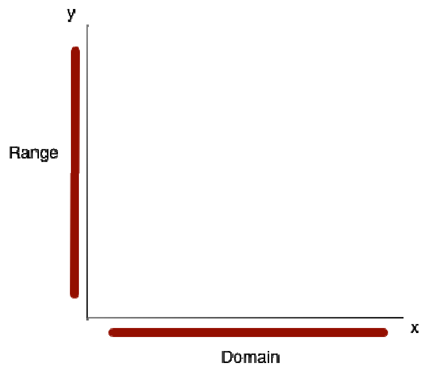
Chapter 4 Functions

Defining terms

One variable is a function of another if the first variable depends upon the second. Hence, a definite rule defines the relationship between the two variables. The independent variable is the variable that causes change in other variables. The dependent variable is the variable whose value depends on the value of other variables.

In the case of the relationship $y = f(x)$, x is the independent variable and y is the dependent variable. Hence the value of y will depend on the value of x.

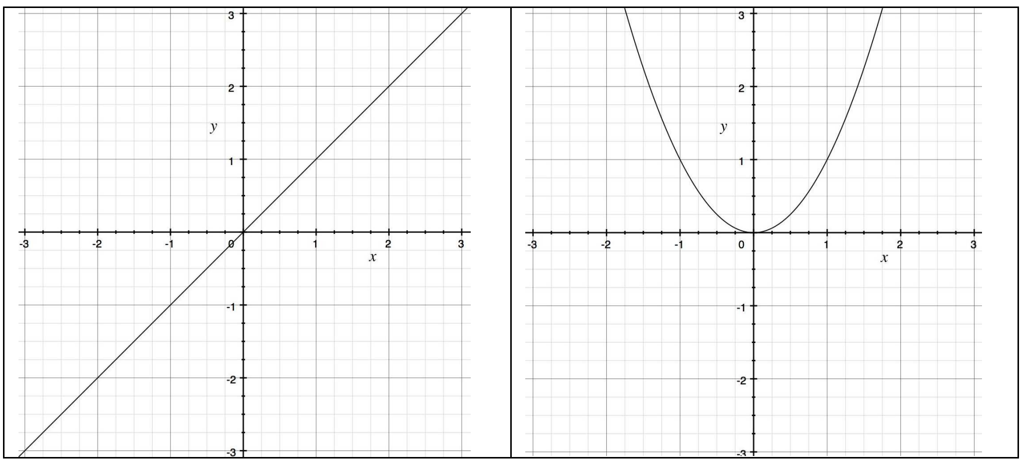
The *domain* of the function f as shown above is the set of all possible values for x (the independent variable). The *range* is the set of the corresponding values of y (the dependent value). To conclude, the range of a function depends on its domain.

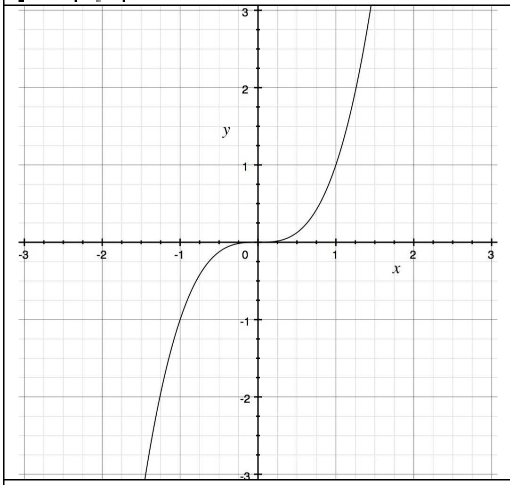
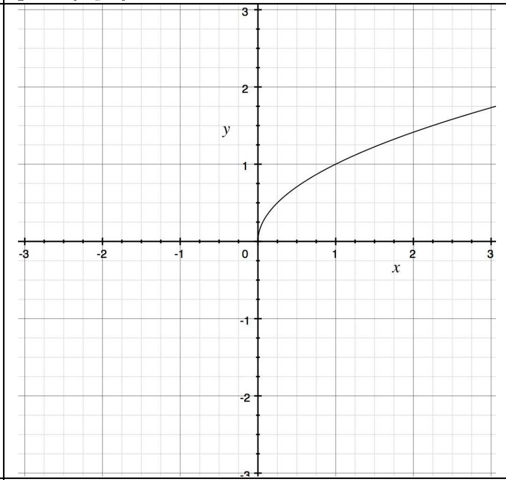
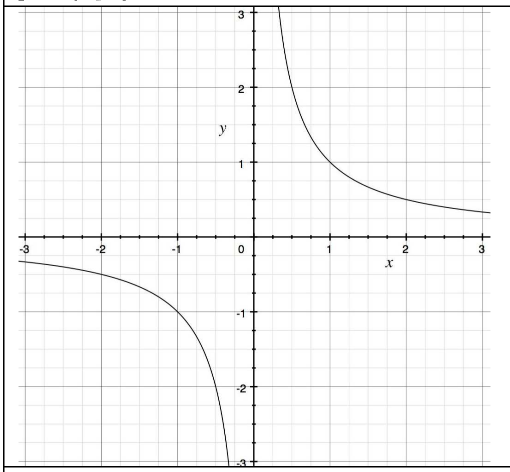
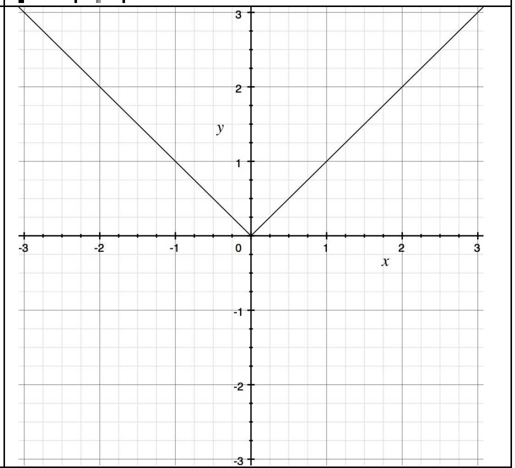


The $f(x)$ can simply be replaced by a different letter, $R(x)$ for example. However, $R(x)$ remains a function, and thus will also have the following property. The natural domain of the function f is the set of all real numbers and these real numbers must give only one y value for each x value.

Graphing functions

The xy plane as sketched above is also called the coordinate system. There are some elementary functions and matching graphs that you should know:



$y = f(x) = x$	$y = f(x) = x^2$
	
$y = f(x) = x^3$	$y = f(x) = \sqrt{x}$
	
$y = f(x) = \frac{1}{x}$	$y = f(x) = \lvert x \rvert$

Linear functions

A linear function is written as $f(x) = ax + b$, where a and b are constant parameters. The properties of this function are:

- The graph is linear, a straight line
- a represents the slope
- b is the intercept on the Y-axis.

To find the equation of a straight line we can use 3 other methods:

1. *Point slope* formula: $y - y_1 = a(x - x_1)$. Use this formula when a point and a slope are known. For example: a line passes through the coordinates (2, 4) and its slope is 4. Find the equation of the line. The information given to us is: $(x_1, y_1) = (2,4)$ and $a = 4$. Thus, $y - 4 = 4(x - 2)$, $y = 4x - 4$.

2. Point-point formula: Use this formula when two points are given. For example, a line passes through (-3, -4) and (2,1). Find the equation of the line. To calculate the slope we can use the following formula: $a = \frac{y_2 - y_1}{x_2 - x_1}$. Note that x_1 cannot be equal to x_2 since the denominator would be equal to zero. Therefore, $a = \frac{1-(-4)}{2-(-3)}$, and $a = \frac{2}{3}$. Then we can proceed constructing the equation: $y - (-4) = \frac{2}{3}[x - (-3)]$, and $2 = \frac{2}{3}x - y$.
3. *Graphical approach*: Another method used with linear equation is graphing- you can graph, for instance, two equations and then find their intersection point.

Linear models and their applications in economics

We discuss two examples of applications of linear functions in economics.

1. The consumption function: $C = a + bY$. In this case, C represents consumption, b represents the marginal propensity to consume (for instance, when your income increases, how much are you going to spend of that increase?) and Y represents (national) income.

2. Supply and demand: basically you can have an equation for demand and an equation for supply (both equations will include a P for price). If you want to find the equilibrium price and the equilibrium quantity, you set the demand and supply equation equal to eachother, and then you simply solve for P to find the equilibrium price. By subsituting the equilibrium price back into one of the equations, you can find the equilbirum quantity as well. For example: take the following two equations, supply and demand respectively, $C = 100 + 20P,\ D = 80 + 40P$. Thus if you set both equations equal to eachother, you find that the equilibirum price equals 1. If you substitute 1 back into one of the equations, for instance $D = 80 + 40(1)$, you find that the equilibrium quantity will be 120.

Quadratic functions

The general quadratic equation is $f(x) = ax^2 + bx + c,$ the graph of the equation is called a parabola. The parabola is always symmetric about the axis of symmetry, the minimum or maximum of the graph. This point is also called the *vertex* of the parabola.

To find this point we can use a shortcut:

- If $a > 0$, then $f(x) = ax^2 + bx + c$ has its *minimum* at $x = -\frac{b}{2a}$
- If $a < 0$, then $f(x) = ax^2 + bx + c$ has its *maximum* at $x = -\frac{b}{2a}$

Optimization using derivatives is explained in later, in chapter 6.

Polynomials

The next step is to consider *cubic functions* of the form $f(x)ax^3 + bx^2 + cx + d$. This is more complicated because the shape of the graphs depends strongly on the coefficients a,b,c and d. In fact, linear, quadratic and cubic functions are all examples of *polynomials.* A general polynomial of *degree n* is defined by:

$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where a's are constants and unequal to zero.

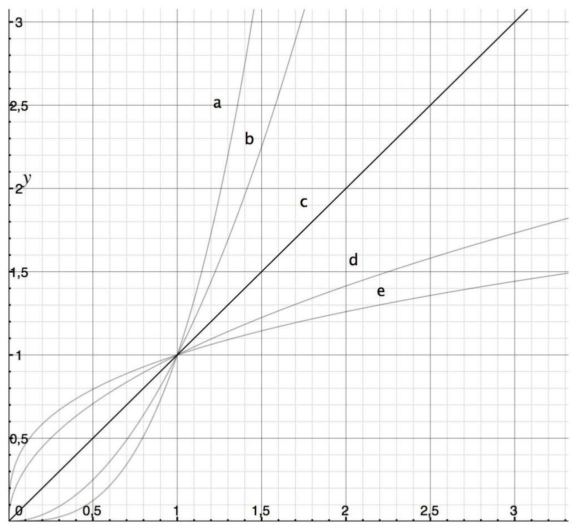
In mathematics problems are often formulated such that a polynomial is equated to zero and to find all the possible solutions.

$$x + 2 \mid x^4 + 2x^3 + 2x^2 + 9 \mid x^3 + 2x - 4$$
$$(-)x^4 (-)2x^3$$
$$+2x^2 \quad +9$$
$$(-)2x^2 (-)\, 4x + 0$$
$$-4x + 9$$
$$(-)4x\ (+)\ 8$$
$$17$$

It is possible to divide polynomials in the following way. Take for example the division $(x^4 + 2x^3 + 2x^2 + 8) \div (x + 2)$. Thus result has to be $x^3 + 2x - 4$, and the remainder is 17. Have a look at the calculation:
A rational function is a function $R(x) = P(x)/Q(x)$ that can be expressed as the ratio of two polynomials. It is only defined if $Q(x) \neq 0$.

Power functions

A power function is defined by $fx=Axr$, $x > 0$ and r and A are constants. x^r can be defined for all rational numbers r, that is for all fractional exponents. In the case of irrational numbers, we can take an approximation (for example of the number π) to make sure that x^r is defined. The shape of the graph also depends on the value of r. See the six planes above for an indication.



The five equations are respectively:

a. $f(x) = x^3$

b. $f(x) = x^2$

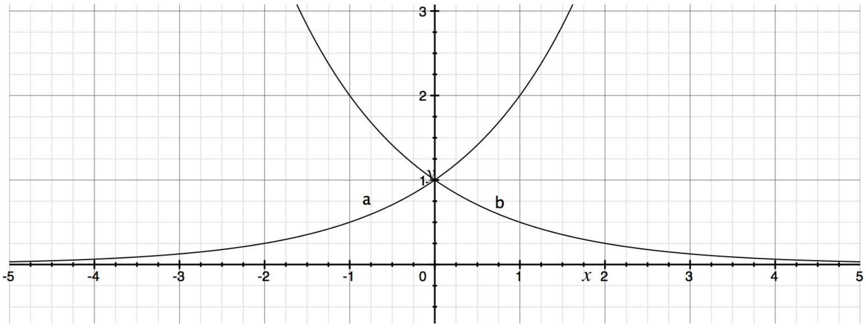
c. $f(x) = x^1$

d. $f(x) = x^{\frac{1}{2}}$

e. $f(x) = x^{\frac{1}{3}}$

Exponential functions

An exponential function is a function where the variable is the power: $f(x) = Aa^x$, where A and a are positive constants. The shape of the graphs depends on the value of a. See for example the two graphs below, in which a is $f(x) = 2^x$, and b is $fx=12x$. As you can see the functions are asymptotic, they do not reach zero or negative values.



Three examples of applications of exponential fuctions are the following:

1. Population Growth: $P(t) = population\,in\,base\,year \times (1 + \frac{rate\,of\,growth}{100})^{time\,in\,years}$

2. Compound Interest: A = $A = P\left[1 \pm \frac{r}{100}\right]^t$ where A = Total Amount, P = Initial amount, r = rate of change/interest rate (% per year in absolute terms), and t = Time in years.

3. Continuous Depreciation: When the value of the asset decreases with the same percentage each year then this process is called continuous depreciation, calculated in the following way:

   $V(t) = P_i\left(1 - \frac{r}{100}\right)^t,$

   $where\ V(t) is\ value\ of\ the\ assest\ on\ t\ year,$

   $P_i\ is\ the\ purchased\ price\ of\ the\ asset$

   $r\ is\ the\ rate\ of\ depreciation$

   $t\ is\ the\ time\ in\ years$

The *natural exponent* function is the function $f(x) = e^x$. The base of this function is the irrational number e and it has turned out to be the most important base for exponential functions.

Logarithmic functions

If $e^u = a$, then we call u the *natural logarithm* of a. The general form of a logarithmic function is therefore $e^{lnx} = x, for\ all\ positive\ values\ of\ x$.

There are some rules for working with the natural logarithm:

1. $\ln(xy) = \ln x + \ln y$
2. $\ln\frac{x}{y} = \ln x - \ln y$
3. $\ln x^p = p\ln x$
4. $\ln 1 = 0$
5. $\ln e = 1$
6. $x = e^{\ln x}, x > 0$
7. $\ln e^x = x$

These rules also apply to logarithms with bases other than e. Any number a can be the base of a logarithm.

The graphs of the exponential function with base e, and the function of the natural logarithm are the following (respectively a and b):