Chapter 5 Functions continued

Shifting graphs

This graphs discussed how the graph of a function $y = f(x)$ can be transformed and therefore how it relates to the graphs of the functions $f(x) + d$, $f(x+d)$, $df(x)$, and $f(-x)$. d can be a positive or negative constant. For an illustration of the following cases, have a look at page 128.

1) $y = f(x) + d$: The graph moves upwards by d units if d is positive, and the graph will move down by d units if d is negative. The new graph is parallel to the first one.
2) $y = f(x+d)$: The graph moves d units to the left if d is positive, and the graph will move to the right by d units, if d is negative.
3) $y = df(x)$: The graph is stretched vertically if d is positive, and if d is negative, the graph will be stretched vertically and will be reflected about the x-axis.
4) $y = f(-x)$: The graph is reflected about the y-axis, as if the y-axis would be a mirror.

Introducing different types and properties of functions

The *sum* of two functions $f(x)$ and $g(x)$ is $F(x) = f(x) + g(x)$. $F(x) = f(x) - g(x)$ is called the *difference* between the two functions. A relevant example of a difference is the profit function, which is the difference between the revenue function and the cost function, $\pi(Q) = R(Q) - C(Q)$.

The *product* of two functions is $h(x) = f(x) \cdot g(x)$. And the *quotient* is $h(x) = \frac{f(x)}{g(x)}$.

A *composite* function is $y = f(g(x))$. $g(x)$ is called the *kernel* or *interior function*, while $f(...)$ is called the *exterior* function. First g applies on x, and on the result applies f.

An example of a composite function:

$f(x) = x^2 - 4, and\ g(x) = x - 2, find\ f(g(x))$.

$f(g(x)) = (x-2)^2 - 4$

$f(g(x)) = x^2 - 4x$

When a graph is symmetric about the y-axis, then $f(x)$ is called an *even* function. When the graph is symmetric about the origin, then $f(x)$ is called an *odd* function. A graph can also be symmetric about the line $x = a$, then $f(x)$ is said to be *symmetric about* a.

Inverse functions

An inverse function is a function that is the reverse of the given function, so if $f(x) = y\ then\ f^{-1}(y) = x$.

For example: The demand for crisps is described by the following function: $D(p) = 20 + 30p$. However, as a producer of the crisps I do not want to know the demand for a specific price, but I want to decide on a certain output, and see what the price of that output will be. Therefore, I want to find the inverse of the demand function.

1. Rewrite the function as an equation: $D = 20 + 30p$.
2. Solve for p: $p = \frac{D-20}{30}$.
3. Switch p and D back and you have the inverse: $D^{-1}(p) = \frac{p-20}{30}$.

More formally, if and only if f is one-to-one, it has an inverse function g with domain B and range A. The function of g is given by the following rule: for each y in B, the value $g(y)$ is the unique number x in A such that $f(x) = y$. Then,

$$g(y) = x \iff y = f(x)$$
$$(x \in A, y \in B)$$

One-to-one means that the equation passes the vertical line test, meaning that, when drawing a vertical line on a certain value on the x-axis, the line can cross the function only once. In other words, for every x exists just one y. Vice versa is not true. If the graph does not pass the vertical line test, then the graph does not represent a function.

The domain of the inverse is the range of the original function, and the range of the inverse is the domain of the original function.

When two functions are inverses of each other, then the graphs of the two are symmetric about the line $y = x$.

General functions

Generally speaking, a function is a rule which to each element in a set A associates one and only one element in a set B. So if we denote the function by f, then the set A is called the domain of f, and set B is called the target or codomain of f. The definition of a function therefore needs to specify the domain, the target and the rule. Sometimes the words transformation and map, or mapping, are used for the same concept as a function.

The particular value of $f(x)$ is called the *image* of the element x. the *range* is the set of elements in B that are images of at least one element in A. The range is therefore a subset of the target.