Chapter 6 Derivatives

Slopes

When studying graphs we are usually interested in the slope of the graph. The steepness of a curve at a particular point can be defined as the slope of the tangent (a straight line that touches the curve at a certain point) to the curve at that specific point. The slope of the tangent to a curve at a particular point is the called the *derivative* of $f(x)$.

The definition of the derivative is the function f at point a, denoted by $f'(x)$, and is given by the formula:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

The equation for the tangent to the graph of $y = f(a)$ at the point $(a, f(a))$ is:

$$y - f(a) = f'(a)(x - a)$$

Finding the derivative

To find the value of the tangent you need to find the derivative and then substitute the given point a. The standard recipe for calculating the derivative $f'(a)$ is the following:

a) Add h to a and compute $f(a+h)$.
b) Compute the corresponding change in the function value: $f(a+h) - f(a)$
c) For $h \neq 0$, form the Newton quotient $\frac{f(a+h)-f(a)}{h}$.
d) Simplify the fraction and cancel h from the numerator and denominator as much as possible.
e) And then you find $f'(a)$ as the limit of this fraction as h tends to 0.

For example, to find $f'(x)$ when $f(x) = 4x^2$ by using this definition (the Newton quotient) of derivatives, you need to follow this procedure:

$$\frac{4(x+h)^2 - 4(x)^2}{h} = \frac{8hx + 4h^2}{h} = 8x + 4h$$

Since h approaches 0 (thus $4h$ will approach 0), the derivative will simply be $f'(x) = 8x$.

There are different ways of writing the derivative. The Leibniz notation, also called the *differential notation* is the following:

$$\frac{dy}{dx} \text{ or } \frac{df(x)}{dx} \text{ or } \frac{d}{dx} f(x)$$

Increasing or decreasing

The definitions of increasing and decreasing functions are the following:

- If $f(x_2) \geq f(x_1)$ whenever $x_2 > x_1$, then f is *increasing* in interval I.
- If $f(x_2) > f(x_1)$ whenever $x_2 > x_1$, then f is *strictly increasing* in interval I.
- If $f(x_2) \leq f(x_1)$ whenever $x_2 > x_1$, then f is *decreasing* in interval I.
- If $f(x_2) < f(x_1)$ whenever $x_2 > x_1$, then f is *strictly decreasing* in interval I.

When we relate this definition of increasing and decreasing to the derivative, we have the following propositions:

- $f'(x) \geq 0$ for all x in the interval I $\Leftrightarrow$ f is increasing in I
- $f'(x) \leq 0$ for all x in the interval I $\Leftrightarrow$ f is decreasing in I
- $f'(x) = 0$ for all x in the interval I $\Leftrightarrow$ f is constant in I

This makes sense since the derivative represents the slope of a function, and thus if a slope is for instance 2, the line will rise from left to right: the function is increasing.

For examine, consider the function $f(x) = 2x^2 - 2$ and define where the function is increasing or decreasing.

1. Find the derivative: $f'(x) = 4x$.
2. Set the derivative equal to zero, and solve for x. Hence, $x = 0.$
3. Substitute a value lower than 0 (for instance -2) into the first derivate, and a value higher than 0 (for instance 2). This will allow you to find the intervals for which the function will be increasing or decreasing:

$f'(x) = 4(-2)$	$f'(x) = 4(2)$
-8 = decreasing	8= increasing

The conclusion is that $f(x)$ is increasing when $x > 0$ and decreasing when $x < 0$.

Derivatives as rates of change

A derivative measures change, to be precise the average rate of change of f over the interval from a to a+h. The slope of the tangent line at a particular point is the *instantaneous rate of change* and is therefore $f'(a)$. The *relative rate of change* of f at point a is $\frac{f'(a)}{f(a)}$. This measure can be used to describe, for instance, how much a variable changed this year (written as a percentage). The relative rate of change is sometimes called the *proportionate rate of change*.

In economics the word *marginal* is widely used to indicate a derivative. For example the marginal propensity to consume and the marginal product are both measures of change.

Limits

Writing $\lim_{x\to a} f(x) = A$ means that we can make $f(x)$ as close to A as we want for all x sufficiently close to (but not equal to) a. Or in other words we can say that $f(x)$ has the number A as its limit, as x tends t a.

There are some important rules for limits. If $\lim_{x\to c} f(x) = C$ and $\lim_{x\to d} g(x) = D$, then:

1) $\lim_{x\to c}(f(x) \pm g(x)) = C \pm D$
2) $\lim_{x\to c}(f(x) \cdot g(x)) = C \cdot D$
3) $\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{C}{D}$, if $D \neq 0$
4) $\lim_{x\to c}(f(x))^r = A^r$ (if A^r is defined and r is any real number)

Differentiation

Differentiation is a process to find the derivative. If a limit exists, then a function is differentiable at x. If the function is a constant, then its derivative is 0. Also additive constants disappear in the process. Multiplicative constants are however preserved.

The basis of differentiation is the *power rule:*

$$f(x) = x^a \rightarrow f'(x) = ax^{a-1}$$

In case of the *sum or difference* of two functions, differentiation is still possible:

$$F(x) = f(x) \pm g(x) \rightarrow F'(x) = f'(x) \pm g'(x)$$

However, when we have to differentiate a product of two functions, we need to use the *product rule:*

$$F(x) = f(x) \cdot g(x) \rightarrow F'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

To illustrate this product rule, consider this example. Find the derivative of $F(x) = (x^2 - 1) \cdot (x + 4)$. The process is the following:

1. $f(x) = (x^2 - 1)$ $\quad f'(x) = 2x$, by using the power rule.
2. $g(x) = (x + 4)$ $\quad g'(x) = x$, by using the power rule.
3. $F'(x) = (2x)(x + 4) + (x^2 - 1)(x)$, by using the product rule.
4. conclusion: $F'(x) = x^3 + 2x^2 + 7x$

In case of a division of two functions, we need to use the *quotient rule:*

$$F'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g^2}$$

Again, we illustrate this rule by an example. Find the derivative of $F(x) = \frac{2x^2+1}{4x-2}$

1. $f(x) = 2x^2 + 1$, $f'(x) = 4x$
2. $g(x) = 4x - 2$, $g'(x) = 4$
3. Conclusion: $F'(x) = \dfrac{4x \cdot (4x-2) - (2x^2+1) \cdot 4}{(4x-2)^2}$

In case of a composite function, hence $f(u(x))$, we need to use the *chain rule*

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

This implies that if y is a differentiable function of u, and u is a differentiable function of x, then y is a differentiable function of x. When $y = u^a$ then
$y' = au$
$a-1$
u'.
U can be considered as a separate function dependent on x.

Formulated formally, if g is differentiable at x_0 and f is differentiable at $u_0 = g(x_0)$, then $F(x) = f(g(x))$ is differentiable at x_0, and

$$F'(x_0) = f'(u_0)g'(x_0) = f'(g(x_0))g'(x_0)$$

So to differentiate a composite function, first differentiate the exterior function w.r.t. the kernel, and then multiply by the derivative of the kernel.

Higher order derivatives

$f'(x) = first\ derivative$
$f''(x) = second\ derivative$

The procedure for finding the second derivative is based on the same properties as explained above, w.r.t. the first derivative. For example, find $f'(x)$ and $f''(x)$ for $f(x) = 4x^3 - 2x + 1$. The answers are $f'(x) = 12x^2 - 2$ and $f''(x) = 24x$.

We can find the second derivative, but we can also find for instance the tenth derivative; the n^{th} derivative is written in the following way: $y^n = f^n(x)$.

Convexity and concavity

1) $f\ is\ convex\ on\ the\ interval\ if\ f''(x) \geq 0\ for\ all\ x'\ s\ on\ the\ interval$
2) $f\ is\ concave\ on\ the\ interval\ if\ f''(x) \leq 0 for\ all\ x's\ on\ the\ interval$

The difference between convex and concave will be important when using economic models, but we will discuss this topic later in detail.

Exponential Function

The exponential function is a particular function with particular properties. It is differentiable, strictly increasing and convex.

The following properties hold for all exponents s and t:

a) $e^s e^t = e^{s+t}$
b) $\frac{e^s}{e^t} = e^{s-t}$
c) $(e^s)^t = e^{st}$

Moreover:

a) $e^x \longrightarrow 0$ as $x \longrightarrow -\infty$
b) $e^x \longrightarrow \infty$ as $x \longrightarrow \infty$

To differentiate an exponential function use the following rule:

$$f(x) = e^x \rightarrow f'(x) = e^x$$

However, if you have to find the derivative of $f(x) = e^{x^2}$, you have to use the chain rule:

$$f(x) = e^u \rightarrow f'(x) = u' \cdot e^u$$

To differentiate other exponential functions we have to start from the following formula:
$a^x = (e^{\ln a})^x = e^{(\ln a)x}$

Therefore when $y = a^x$ then $y' = a^x \ln a$

Logarithmic Functions

The natural logarithmic function $g(x) = \ln x$ is differentiable, strictly increasing and concave in (0,∞).

By definition $e^{\ln x} = x$ for all $x > 0$, and $\ln e^x = x$ for all x. The following properties hold for all positive x and y.

a) $\ln(xy) = \ln x + \ln y$
b) $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$
c) $\ln(x^p) = p \ln x$

To differentiate a natural logarithmic function, use the following formula:

$$f(x) = \ln x \rightarrow f'(x) = \frac{1}{x}$$

For example, the derivative of $f(x) = \ln x + x^2$ is $f'(x) = \frac{1}{x} + 2x$.