Chapter 7 Application of Derivatives:

Implicit Differentiation

1. Implicit Differentiation is used when the function is not in the form a= f(b), but the variables x and y could be on either side of the equation.
2. Instead of getting the equations to the form a=f(b), just take derivative on both sides (REMEMBER: re-write 'a' as f(b) considering 'a' to be a function of 'b')
3. Solve the equation for y'

For example, consider the equation $2a^2 + 4ab = 2b^2$. To be able to differentiate, consider it as $2a^2 + 4af(b) = 2[f(a)]^2$. Remember that we have to use the chain and product rule where applicable:

$$2 \times 2 \times a + 4a^0 f(a) + 4af'(a) = 2 \times 2 \times f(a) \times f'(a)$$
$$4a + 4f(a) + 4af'(a) = 4f(a)f'(a)$$
$$4a + 4f(a) = f'(a)[4f(a) - 4a]$$
$$f'(a) = \frac{4a + 4f(a)}{4f(a) - 4a} = \frac{4a + 4b}{4b - 4a} = b'$$

In economics implicit differentiation is widely used, because many economic problems are formulated in a system of implicit equations relating different variables.

Inverse and Differentiation

If a function is differentiable, and it is strictly increasing or decreasing then for that interval the function will have an inverse function (b in the formulation below). Formally:

$$for\ a\ function\ b = f(a), \qquad h'(b_i) = \frac{1}{f'(a_i)}$$

For example, if we want to find the inverse of the function $f(x) = e^x$, then we need to follow this reasoning: $y = e^x = f(x)$ and therefore the inverse is $x = g(y) = lny$.To find the derivative we use this rule: $g'(y) = \frac{1}{f'(x)}\ from\ the\ equation\ \ given\ above$. And the result is:

$$\frac{1}{y} = \frac{1}{e^x} \qquad therefore, the\ inverse\ function\ is\ e^x = y$$

Understanding Linear Approximations

When there is a complex function then we can sometimes replace it with a linear function with a similar graph. This is called a linear approximation to function $f(x)$ about a certain point $x = a$. The formula for the approximation is:

$f(x) \approx f(a) - f'(a)(x - a)$, when x is close to a.

For example, to find the approximate value of $(1.003)^{60}$, consider this equation to be $f(x) = x^{60}, where\ x = 1.003.$ $Now\ let\ us\ assume\ x = 1.003 \approx a = 1$

By using the formula we find $f(1.003^{60}) \approx f(1) + f'(1^{60})(1.003 - 1)$. Then we solve the part $f'(1^{60}) = 60 \times 1^{59}$ = 60. And therefore, $f(1.003^{60}) \approx 1 + 60(0.003) = 1.018$.

In general terms, when we consider a differentiable function $f(x)$, then the expression $f'(x)dx$ is called the *differential* of $y = f(x)$, and its denoted by dy. So, the formal expression is $dy = f'(x)dx$. Here dx is a single symbol representing the change in the value of x and it is not a multiplication of d and x.

The following rules hold:

1. $d(uf + vg) = udf + vdg$

$(where\ d\ stands\ for\ differentiate, \quad u, v\ are\ constants, \quad f, g\ are\ functions)$

2. $d(fg) = gdf + fdg$
3. $d\left(\frac{f}{g}\right) = \frac{gdf - fdg}{g^2}$

Understanding Polynomial Approximations

When there is a complex polynomial then we can sometime replace it by an approximation to make calculations easier. For example a quadratic approximation:

For a fixed value $a = u$

$$f(a) \approx f(u) + f'(u)(a-u) + \frac{1}{2} f''(u)(a-u)^2$$

Other approximations of polynomials with a higher order follow the same logic:

$$f(a) \approx f(u) + \frac{f'(u)}{1!}(a-u) + \frac{f''(u)}{2!}(a-u)^\wedge 2 + \ldots\ldots\ldots + \frac{f^n(u)}{n!}(a-u)^n$$

Taylor's formula

One of the main results of mathematical analysis in economics is called Taylor's formula and is a remedy for the problem of the difference between the real formula and an approximation. The difference between the two is called the *remainder*. Therefore, by definition:

$n{+}1(x)$

The explicit formula for the remainder R is given by the following proposition, called the Lagrange form:

$$R_{n+1}(x) = \frac{1}{(n+1)!} f^{(n+1)}(c) x^{n+1}$$

Based on the supposition that a function is n+1 times differentiable in an interval excluding x and 0 t (the number c is also contained in this interval).

Using this formula we get to *Taylor's formula:*

$$f(x) \approx f(0) + \frac{1}{1!}f'(0)x + \cdots + \frac{1}{n!}f^{(n)}(0)x^n + \frac{1}{(n+1)!}f^{(n+1)}(c)x^{n+1}$$

Elasticities

The *price elasticity of demand* measures by what percentage the quantity demanded changes when the price increases by 1%. The symbol used for price elasticity is $EL_p\, D(p)$. The price elasticity with respect to demand can be calculated using the formula:

$$EL_p\, D(p) = \frac{p}{D(p)} \times \frac{dD(p)}{dp}$$

The general formulation of the elasticity of a function with respect to the variable x is:

$$EL_x f(x) = \frac{x}{f(x)} \times f'(x)$$

For example, find the price elasticity using the $D(p) = \frac{p-1}{p+1}$ by the following procedure:

1. $EL_p D(p) = \frac{p}{D(p)} \times D'(p)$ is the general formulation of the solution.
2. $EL_p D(p) = \frac{p\,(p+1)}{(p-1)} \times \frac{(p+1)-(p-1)}{(p+1)^2}$. $D'(p)$ can be found using the quotient rule.
3. $EL_p D(p) = \frac{p}{(p-1)} \times \frac{2}{p+1}$ is given by rewriting and therefore, the price elasticity is $EL_p D(p) = \frac{2p}{(p-1)^2}$.

Continuity

Graphically speaking, a function is said to be continuous if its graph is connected, that is, it has no breaks. Otherwise a function is called discontinuous. In terms of limits, a function is continuous at point $x = a$ if:

$$\lim_{x \to a} f(x) = f(a)$$

Therefore, three conditions need to be fulfilled, otherwise a function is called discontinuous:

1. The function must be defined at $x = a$.
2. The limit of $f(x)$ as x tends to a needs to exist.
3. This limit must be exactly equal to $f(a)$.

The four properties of continuous functions are the following. If f and g are continuous at a, then:

1. $f + g$ and $f - g$ are continuous at a
2. fg and $\frac{f}{g}$ are continuous at a
3. $[f(x)]^r$ is continuous at a if $[f(a)]^r$ is defined
4. If f is continuous and has an inverse on the interval I, then its inverse f^{-1} is continuous on the same interval.

Any function that can be constructed from continuous functions by addition, subtraction, multiplication, division and composition is also continuous at all points where it is defined.

More on limits

We say that a limit exists if it tends towards a number a, sufficiently close. If a limit does not exist, we say that it tends to infinity. In that case a *vertical asymptote* exists.

In some cases the value of a limit depends on from which side you start reasoning. The notation of one sided limits, respectively from below or from above, is as follows:

$\lim_{x \to a^-} f(x) = B \text{ or } f(x) \to B \text{ as } x \to a^-$
and

$\lim_{x \to a^+} f(x) = A \text{ or } f(x) \to A \text{ as } x \to a^+$

Now we can define one sided continuity. In the first case we say that the function is *left continuous* and in the second case we say that the function is *right continuous*.

We can also define the limit of a function when x tends to infinity. Then encounter horizontal asymptotes, and we write:

$\lim_{x \to \infty} f(x) = A \text{ or } f(x) \to A \text{ as } x \to \infty$

When studying composites of functions with respect to continuity we have to consider the following properties, especially the last two:

If $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to a$, then:

- $f(x) + g(x) \to \infty$
- $f(x)g(x) \to \infty$
- $f(x) - g(x) \to ?$
- $f(x)/g(x) \to ?$

The relationship between continuity and differentiability is relatively simple. If a function is differentiable at $x = a$ then the function is continuous at $x = a$. However, if there is a kink in the graph (and function) at point a, then the function is continuous but not differentiable at that point.

The conclusion of this discussion about continuity of functions is that, if a function is continuous in a closed interval $[a, b]$, then according to the *intermediate value theorem*:

1. If $f(a)$ and $f(b)$ have different signs, then there is at least one c in the interval such that $f(c) = 0$.
2. If $f(a) \neq f(b)$, then for every intermediate value y in the open interval between $f(a)$ and $f(b)$ there is at least one c such that $f(c) = y$.

The intermediate value theorem is normally used to state that an equation $f(x) = 0$ has a solution in a given interval. *Newton's method* also approximates the location of the zero. The method generates a sequence of points that converges to a zero quickly, given by the formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Formally we say that a sequence is converging to a number s if s_n can be made arbitrarily close to s by choosing n sufficiently large: $\lim_{n\to\infty} s_n = s$. In the opposite case a sequence is said to diverge.