Chapter 8 Optimal Points

Extreme points

The extreme points of a function are where it reaches its largest and its smallest values, the maximum and minimum points. Formally,

- $c \in D$ is the maximum point for $f \Leftrightarrow f(x) \leq f(c)$ for all $x \in D$
- $d \in D$ is the minimum point for $f \Leftrightarrow f(x) \geq f(d)$ for all $x \in D$

Where the derivative of the function equals zero, $f'(x) = 0$, the point x is called a *stationary point* or *critical point*. For some point to be the maximum or minimum of a function, it has to be such a stationary point. This is called the *first-order condition*. It is a necessary condition for a differentiable function to have a maximum of minimum at a point in its domain.

Stationary points can be local or global maxima or minima, or an inflection point. We can find the nature of stationary points by using the first derivative. The following logic should hold:

- If $f'(x) \geq 0$ for $x \leq c$ and $f'(x) \leq 0$ for $x \geq c$, then $x = c$ is the maximum point for f.
- If $f'(x) \leq 0$ for $x \leq c$ and $f'(x) \geq 0$ for $x \geq c$, then $x = c$ is the minimum point for f.

Is a function concave on a certain interval I, then the stationary point in this interval is a maximum point for the function. When it is a convex function, the stationary point is a minimum.

Economic Applications

Take the following example, when the price of a product is *p*, the revenue can be found by $R = 5p^2 + 20p + 16$. What price maximizes the revenue?

1. Find the first derivative: $R' = -10p + 20$
2. Set the first derivative equal to zero, $0 = -10p + 20$, and solve for p.
3. The result is $2 = p$.

Before we can conclude whether revenue is maximized at 2, we need to check whether 2 is indeed a maximum point. The easiest way is to find the second derivative, and if the second derivative is negative for 2, then 2 is indeed a maximum point. The second derivative will be positive if 2 is a minimum point. The second derivative is $R'' = -10$. Thus, 2 is indeed the maximum point.

Extreme Value Theorem

The reasoning above is mainly based on the extreme value theorem: Suppose that f is a continuous function in a closed and bounded interval $[a, b]$. Then there exists a point d in $[a, b]$ where f has a minimum, and a point c in $[a, b]$ where f has a maximum, so that:

$$f(d) \leq f(x) \leq f(c), \text{ for all } x \text{ in } [a, b]$$

Every extreme must be one out of three options:

- Interior points where $f'(x) = 0$
- End points of the closed and bounded interval
- Interior points where f' does not exist

Another example: Find the maximum and minimum values of the function
$f(x) = 2x^2 - 4x + 5$ for the interval $x \in [0,5]$.

1. Find the stationary points using the first order condition:
   a. $f'(x) = 4x - 4$, $0 = 4x - 4$, $x = 1$
   b. $f(1) = 2(1)^2 - 4(1) + 5$, $f(1) = 3$
2. Find the y-values of the end points of the interval:
   a. $f(0) = 2(0)^2 - 4(0) + 5$, $f(0) = 5$
   b. $f(5) = 2(5)^2 - 4(5) + 5$, $f(5) = 35$
3. By looking at all three points (0,5) (1,3) (5,35) we can see that (1,3) is the minimum value.

Local maximums and minimums

Although usually the global maxima or minima are of interest for economists, also local extreme points are sometimes relevant. Formally the function f has a local, or relative, maximum at c if there exists an interval (α, β) about c such that $f(x) \leq f(c)$ for all x in (α, β) which are in the domain of f.

To find local extreme points the first-derivative has to be used as well. The following reasoning holds:

- If $f'(x) \geq 0$ throughout some interval to the left of point c and $f'(x) \leq 0$ to the right of the point, then $x = c$ is a local maximum point for f.
- If $f'(x) \leq 0$ throughout some interval to the left of point c and $f'(x) \geq 0$ to the right of the point, then $x = c$ is a local minimum point for f.
- If $f'(x) > 0$ throughout some interval both to the left and to the right of point c then $x = c$ is not a local extreme point for f. The same holds for $f'(x) < 0$.

A more formal way to know if the stationary points are local extreme points, is by use of the second derivatives:

- $f'(c) = 0$ and $f''(c) < 0$, then $x = c$ is a strict local maximum point
- $f'(c) = 0$ and $f''(c) > 0$, then $x = c$ is a strict local minimum point
- $f'(c) = 0$ and $f''(c) = 0$, then we do not know the nature of the point

Inflection Points

If in the last case the second derivative f'' changes sign at point c, then that point is called an inflection point. The meaning of an inflection point is that the function f is changing from concave to convex or vice versa. An example of a function with an inflection point is $f(x) = \sqrt[3]{x}$. A function is called concave if the line segment joining any two points on the graph is below or on the graph. For convex functions the reasoning is vice versa. Formally:

$$f''(x) < 0 \; for \; all \; x \in (c, d), f(x) is \; concave \; in \; (c, d)$$
$$f''(x) > 0 \; for \; all \; x \in (c, d), f(x) is \; convex \; in \; (c, d)$$