Chapter 9 Integrals

Indefinite integrals
To reverse the process of differentiation we use integration, or in other words we take the anti derivative. Formally the *indefinite integral* is

$$\int f(x)dx = F(x) + C$$

when, $F'(x) = f(x)$.

The symbol in front is called the integral sign and $f(x)$ is the *integrand*. To indicate that the variable of integration is x, it is written dx. It is called the indefinite integral because $F(x) + C$ should be seen as a class of functions, all having the same derivative $f(x)$.

By definition the derivative of an indefinite integral equals the integrand:

$$\frac{d}{dx}\int f(x)\,dx = f(x)$$

Thus, integration and differentiation cancel each other out.

Some important formulas directly result from differentiation formulas as stated in other chapters (a is a constant):

- $\int x^a\,dx = \frac{1}{a+1}x^{a+1} + C$
- $\int \frac{1}{x}dx = \ln|x| + C$
- $\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$
- $\int a^x\,dx = \frac{1}{\ln a}a^x + C$
- $\int af(x)\,dx = a\int f(x)\,dx$
- $\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$

We give an example of solving an integration problem. Consider the function $f = 4x^2$ then the derivative is $f'(x) = 8x$. If we take the integral of this derivative we get the indefinite integral: $\int 8x = 4x^2 + c$.

Thus, to integrate you first add a number to the exponent on the variable x and you divide the number in front of x by this new exponent, this rule can be written as: $\frac{1}{n+1}x^{n+1} + C$. Then we need to add an undefined constant, to cover for the possibility of a constant that has disappeared in the differentiation process. To illustrate this, consider the following example: If we need to differentiate the function $f(x) = x^2 + 4x + 2$, then the derivative would be $f'(x) = 2x + 4$. Integrating this function would give us $x^2 + 4x$. However, we have no way of knowing for sure if some constant was part of the original function, in this case 2. That is why we need to add c at the end of the indefinite integral; the c represents any number that could have been behind $x^2 + 4x$.

Definite Integrals

The exact area under a curve (between the graph and the x-axis) is given by the definite integral. The definite integral has a lower and a upper limit of integration. A definite integral is defined in the following way:

$$\int_a^b f(x)dx = |_a^b\ F(x) = F(b) - F(a), where\ F'(x) = f(x) for\ all\ x \in (a, b)$$

For example, to solve the integral $\int_2^4 (x^2 - 4x)dx$ proceed like this:

$\int_2^4 (x^2 - 4x)dx = |_2^4 \left(\frac{1}{3}x^3 - 2x^2\right) = [\frac{1}{3}(4)^3 - 2(4)^2] - [\frac{1}{3}(2)^3 - 2(2)^2] = \frac{16}{3} - \frac{16}{3}$ = 0.

Thus, there is no area under the curve.

If the defined area under $f(x)$ is negative, then the area you are finding is below the x-axis. However, note that you are still finding an area. There is simply a negative side in front of the integral. The area under the x-axis is simply subtracted from the total area.

Properties of Definite Integrals

- $\int_a^b f(x)dx = -\int_b^a f(x)dx$
- $\int_a^a f(x)dx = 0$
- $\int_a^b \propto f(x)dx = \propto \int_a^b f(x)dx$, where α is an arbitrary number
- $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$
- $\int_a^b [\alpha f(x) + \beta g(x)]dx = \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx$

Moreover, the derivative of the definite integral with respect to the upper limit of integration is equal to the integrand (the expression you are integrating) at that limit:

$$\frac{d}{dz}\int_a^z f(x)dx = F'(z) = f(z)$$

The derivative of a definite integral with respect to the lower limit of integration is equal to minus the integrand evaluated at the limit:

$$\frac{d}{dt}\int_z^w f(x)dx = -F'(z) = -f(z)$$

These two rules can be generalized into the following formula:

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x)dx = f(b(t))b'(t) - f(a(t))a'(t)$$

Economic application

An example of an application in economics of the integral is the calculation of the consumer and producer surplus. See page 313 for a graphical illustration. The two formulas are:

$$CS = \int_0^{Q*} [f(Q) - P*]dQ$$

$$PS = \int_0^{Q*} [P* - g(Q)]dQ$$

Integration by Parts

To integrate an equation by parts we need to first re-write the equation in the form of $f(x) \times h'(x)dx$. In words this is a function for the variable x multiplied by the derivative of another function in terms of this variable x. Just as the derivative of a product is not the product of the derivatives, we also need to use a different formula for integration, which is:

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

A similar formula exists for definite integrals:

$$\int_a^b f(x)g'(x)dx = |_a^b f(x)g(x) - \int_a^b f'(x)g(x)dx$$

For example, to solve the indefinite integral $\int 2x^4 dx$, we have to rewrite it to $\int (x)^3 \times 2x$. Then $f(x) = x^3 and\ h(x) = x^2, since\ 2x\ is\ the\ derivative\ of\ x^2$, implying that the formula is:

$$\int x^3 \times 2x(dx) = x^3 \times x^2 - \int 3x^2(dx) \times x^2$$

Therefore, $\int_q^p f(x)h'(x)dx = [_q^p f(x)h(x) - \int_q^p f'(x)dxh(x)$

Integration by Substitution

To integrate by substitution the equation should be re-written as a composite function $f(h(x)) \times h'(x)dx$. Then the rule for integration by substitution is:

$$\int f(g(x))g'(x)dx = \int f(u)du, \text{ where } u = g(x)$$

For complicated integrals of the form $\int G(x)dx$ we can use the following procedure:

1. Pick out a part of $G(x)$ and introduce it as a new variable: $u = g(x)$
2. Compute $du = g'(x)dx$
3. Using the substitution $u = g(x)$, $du = g'(x)$, transform $\int G(x)\,dx$ to $\int f(u)\,du$
4. Then find $\int f(u)\,du = F(u) + C$
5. Replace u by $g(x)$. This gives the final answer $\int G(x)\,dx = F(g(x)) + C$

Infinite Intervals

If a function f is continuous, then the integral $\int_a^b f(x)\,dx$ is defined for all $b \geq a$. If the limit $b \to \infty$ exists, then f is integrable over the infinite interval $[a, \infty)$ and the improper integral is said to converge:

$$\int_a^{\infty} f(x)\,dx = \lim_{b \to \infty} \int_a^b f(x)\,dx$$

If the limit does not exist, then the integral is said to diverge. In the case of the interval $(-\infty, \infty)$, the improper integral is defined as:

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_0^{\infty} f(x)\,dx$$

If both integrals on the right hand side converge, then the entire integral is said to converge. Otherwise, it diverges.

An alternative test for convergence exists. Suppose that f and g are continuous and $|f(x)| \leq g(x)$ for all $x \geq a$. Then $\left|\int_a^{\infty} f(x)\,dx\right| \leq \int_a^{\infty} g(x)\,dx$, and if one converges than the other also converges.

Reviewing Differential Equations

Differential equations are equations where the unknowns are functions, and where the derivatives of these functions also appear. In the case of differential equations, the independent variable is usually denoted as t, because usually time is the independent variable.

To find all functions that solve $\dot{x}(t) = f(t)$ ($\dot{x}$ is the derivative of x with respect to time) we know that the general solution is an indefinite integral:

$$x(t) = \int f(t)\,dt + C$$

More complicated versions of differential equations are often related to growth rates. There are two general types of differential equations, separable and linear differential equations.

A separable differential equation is of the type $\dot{x} = f(t)g(x)$ and the unknown function is $x = x(t)$. To solve we have to take four steps.

1. Write the equation differently: $\frac{dx}{dt} = f(t)g(x)$
2. Separate the variables: $\frac{dx}{g(x)} = f(t)dt$
3. Integrate each side: $\int \frac{dx}{g(x)} = \int f(t)\,dt$
4. To find a solution for the equation in step 1, evaluate both integrals, and rewrite for x if possible.

A first-order linear equation is of the form $\dot{x} + a(t)x = b(t)$ where $a(t)$ and $b(t)$ are continuous functions in a certain interval. $x = x(t)$ is the unknown in the function.