Chapter 1 Review of Basic Mathematics

Numbers

Natural numbers are the basic and familiar numbers, also called positive integers: 1,2,3,4,… Together with the negative integers, -1,-2,-3, …, they make up the *integers.* These integers can be even (2,4,6,…) or odd (1,3,7,...).

Rational numbers are the numbers in the form $\frac{d}{e}$, in which d and e are both integers, and $e \neq 0$. For $e = 0$ the fraction cannot be defined for any real number d. Note that any integer n is a rational number because $n = \frac{n}{1}$.

The most common system of writing numbers is the *decimal system*. It is a system in which 10 is the base number and every natural number can be written using only digits (1,2,3,4,5,6,7,8,9,0). It follows that each combination of digits can be written as a sum of powers of 10. For example:

$$1984.14 = 1 \cdot 10^3 + 9 \cdot 10^2 + 8 \cdot 10^1 + 4 \cdot 10^0 + \frac{1}{10^1} + \frac{4}{10^2}$$

Rational numbers can be *finite decimal fractions* or *infinite decimal fractions.* The latter is the case when the rational number cannot be written by using a finite number of decimal places. For example, $\frac{100}{3} = 33.333 \ldots$

Real numbers are arbitrary infinite decimal fractions, of the form $x = \pm m.\alpha_1\alpha_2\alpha_3 \ldots$. In the case of rational numbers, the decimal fraction will always be recurring or periodic, that is, there is repetition in the decimal expansion ($\frac{11}{70} = 0.157142857142 85 \ldots$). If the decimal fraction is not periodic, then these numbers are called *irrational numbers*.

Powers
A 100 times multiplication of w can be written as w^{100}. W is called the base and 100 is the exponent. In general terms we can say that the expression $w^n = w \cdot w \cdot w \cdot w \cdot \ldots$ is called the n-th power of w.

The properties of powers are the following:

1. $w^0 = 1$
2. $w^a \times w^b = w^{a+b}$
3. $\frac{w^a}{w^b} = w^{a-b}$
4. $\frac{1}{w^a} = w^{-a}$
5. $(w^a)^b = w^{ab}$

6. $\left(\frac{w}{z}\right)^a = \frac{w^a}{z^a}$

7. $(wz)^a = w^a z^a$

Using algebra we can find that in general: $(w+z)^a \neq w^a + z^a$.

To give an example of a practical application, exponents are used to compound interest:

$$A = P\left[1 \pm \frac{r}{100}\right]^t$$

A is the total new amount, P is the initial amount, r is the rate of change, or the interest rate in percentages per year and t is time in years. It is an addition when the rate, r, is increasing, and a deduction when the rate in decreasing. 1+ r100 is called the growth factor for a growth or decline of r%.

A numerical example is the following: Williams wins a price of 500 and he deposits this amount on his bank account that pays 6% interest per year. After 5 years he will have:

$$A = 500\left[1 + \frac{6}{100}\right]^5 = 500[1.06]^5 \cong €669.11$$

Algebra rules

1. $a + b = b + a$
2. $(a + b) + c = a + (b + c)$
3. $a + 0 = a$
4. $a + (-a) = 0$
5. $ab = ba$
6. $(ab)c = a(bc)$
7. $1 \cdot a = a$
8. $a.a^{-1} = 1, a \neq 0$
9. $(-a).b = a.(-b) = -a.b$
10. $(-a).(-b) = a.b$
11. $a(b + c) = a.b + a.c$
12. $(a + b)c = ac + bc$
13. $(a + b)^2 = a^2 + 2ab + b^2$

14. $(a-b)^2 = a^2 - 2ab + b^2$

15. $(a+b)(a-b) = a^2 - b^2$, also called the *difference-of-squares formula*

16. $(a)^{w/z} = \sqrt[z]{a^w}$

Fractions

A fraction is called proper when the numerator is smaller than the denominator, $\frac{3}{9}$, and a fraction is called improper when the numerator is larger than the denominator, $\frac{9}{3}$.

We list the most essential properties of fractions below:

1. $\frac{a \cdot c}{b \cdot c} = \frac{a}{b}$
2. $\frac{-a}{-b} = \frac{a}{b}$
3. $-\frac{a}{b} = \frac{-a}{b}$
4. $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$
5. $\frac{a}{b} + \frac{c}{d} = \frac{a \cdot d + b \cdot c}{b \cdot d}$
6. $a + \frac{b}{c} = \frac{a \cdot c + b}{c}$
7. $a \cdot \frac{b}{c} = \frac{a \cdot b}{c}$
8. $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$
9. $\frac{a/b}{c/d} = \frac{a}{b} \cdot \frac{d}{c}$

The first rule is very important because it can be used to simplify fractions by factoring the numerator and the denominator, or in other words, by cancelling common factors.

Fractional powers

In economics fractions in powers are common. Fractional powers have some important algebraic consequences:

1. $a^{1/2} = \sqrt{a}$, valid only if $a \geq 0$
2. $\sqrt{ab} = \sqrt{a}\sqrt{b}$
3. $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$
4. $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$
5. $a^{\frac{1}{n}} = \sqrt[n]{a}$
6. $a^{\frac{p}{q}} = \left(a^{\frac{1}{q}}\right)^{p} = (a^{p})^{\frac{1}{q}}$

Inequalities

If a number *a* is strictly greater than a number *b*, we write $a > b$. If *a* is greater than or equal to *b,* we write $a \geq b$. If a number *a* is strictly smaller than a number *b*, we write $a < b$. And we write $a \leq b$ when *a* is smaller than or equal to b. There are six fundamental properties related to inequalities:

1. $a > 0$ and $b > 0$ imply $a + b > 0$ and $a \cdot b > 0$
2. If $a > b$, then $a + c > b + c$
3. If $a > b$ and $b > c$, then $a > c$
4. If $a > b$ and $c > 0$, then $ac > bc$
5. If $a > b$ and $c < 0$, then $ac < bc$
6. If $a > b$ and $c > d$, then $a + c > b + d$

These properties remain valid when each $>$ is replaced by a $\geq$.

When the two sides of an inequality are multiplied by a positive number, the direction of the inequality is preserved. But when the two sides are multiplied by a negative number, the direction of the inequality is reversed. For example: when the inequality 30b > 20c is multiplied by -2, the inequality becomes -60b < -40c.

An example of how to solve the inequality $\frac{6a-2}{-5} \leq 10$

$$=> multiplying\ both\ sides\ by -5, \frac{6a-2}{-5} \times -5 \geq 10 \times -5,$$
$$=> 6a - 2 \geq -50,$$
$$=> 6a \geq -50 + 2$$
$$=> a \geq \frac{-48}{6} \ => \ a \geq -8$$

Sign Diagrams

Sometimes it can be useful to use a sign diagram to show all the possible values for an inequality. See pages 25, 26 and 27 for illustrations of such diagrams.

We use the following inequality to illustrate the sign diagram: $\frac{(a-6)}{a-3} > 2 - a$.

1. First bring all everything to one side of the inequality: $\frac{(a-6)}{a-3} - 2 + a > 0$

2. Then, make sure you have a common denominator and solve the numerator: $\frac{a^2-4a}{a-3} > 0$

3. The result can be used to make a sign diagram: $\frac{a(a-4)}{a-3} > 0$

Now for every part of the inequality there will be a horizontal number line, showing for which numbers the inequality holds and for which numbers it does not hold, hence in this case for which number the left-hand side is indeed positive. In this case we need to consider three parts and the whole: a, $a-4$, $a-3$, and $\frac{a(a-4)}{a-3}$.

Intervals

There are four different types of intervals. An interval is a set of numbers that lies between two points on a line. There are four types of *bounded* intervals:

(a, b)	Open interval	$a < x < b$
$[a, b]$	Closed interval	$a \leq x \leq b$
$(a, b]$	Half-open interval	$a < x \leq b$
$[a, b)$	Half-closed interval	$a \leq x < b$

Apart from these bounded intervals, an interval can also be unbounded when there is no upper limit. This happens in the case of infinity, for example in the case of the following interval: $[a, \infty)$. An interval going to infinity can never be completely closed.

Absolute value

The absolute value of a number *a* is the distance between the real number and zero on a number line. Is *a* positive then the absolute value is the number a itself. Is *a* negative, then the absolute value is actually –*a*. The denotation is as follows:

$$|a| = \begin{cases} a \ if \ a \geq 0 \\ -a \ if \ a < 0 \end{cases}$$

The distance between a and b on the number line $= |a - b| = |b - a|$

And $|x| < a$ means that $-a < x < a$. The $<$ can be replaced by $\leq$.

Chapter 2 Review of basic mathematics, Equations

Solving simple equations

Solving an equation means to find all values of the variables for which the equation is satisfied. If a certain value makes the expression undefined, then that value is not allowed. The trick is to rewrite the equations such that you can find a solution. Hence, first you isolate x, and then you simply solve for x:

Take for example the equation $4x + 5 = 12 - 3x$. The proper procedure to find x is the following:

$$4x - 3x + 5 = 12$$
$$7x + 5 = 12$$
$$7x = 17$$
$$X = 17/7$$

In cases of fractions you will have to find the lowest common denominator in order to solve for x. Take for example the following equation:

$$\frac{4x+5}{x-2} - \frac{9}{x^2-4} = \frac{12}{x+2}$$

We cannot subtract the fractions since the denominators of the fractions are not the same. Therefore we have to rewrite the equation to create the common denominator. In this case the easiest is to choose $x^2 - 4$, since its factor pairs are $(x-2)$ and $(x+2)$. For every fraction we need to multiply the numerator and denominator with the same expression to keep the meaning the same. The algebraic procedure is as follows:

$$\frac{(4x+5)(x+2)}{(x-2)(x+2)} - \frac{9}{(x-2)(x+2)} = \frac{12(x-2)}{(x-2)(x+2)}$$

$$(4x+5)(x+2) - 9 = 12(x-2)$$

$$4x^2 + 13x + 1 = 12x - 4$$
$$4x^2 + x + 5 = 0$$

At this point we would use the quadratic formula to solve for x. However, we will discuss this later on. This example is only meant to illustrate how to get rid of fractions.

Linear equations

The general linear equation is $y = ax + b$, in which y and x are variables, and a and b are called parameters. Parameters can take on different values, but the logic of the equation does not change.

Two examples of linear equations in economics are:

1. $Y = C + \bar{I}$ A country's GDP (Y) equals consumption (C) plus investment ($\bar{I}$) treated as fixed.

2. $C = a + bY$ Consumption (C) is a linear function of GDP (Y)

Together these two are a macroeconomic model. By substituting 2 into 1 we find $Y = a + bY + \bar{I}$ and we can rewrite this into $Y = \frac{a}{1-b} + \frac{1}{1-b}\bar{I}$. This expression gives the *endogenous* variable Y in terms of the exogenous (given) variable $\bar{I}$ and the two parameters. Economists say that the system of two equations is called the structural form and the last equation is called to reduced form, expressing endogenous variables as functions of exogenous variables.

Quadratic Equations

To solve quadratic equations, or second-degree equations we need to find a way to get rid of the powers in the equation. The general quadratic equation is $ax^2 + bx + c = 0, (a \neq 0)$.

Three rules for solving quadratic equations are the following:

1. For this general quadratic equation $ax^2 + bx + c = 0,\ where\ b^2 - 4ab \geq 0\ and\ a \neq 0$

   The solution for x can be found using this formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

2. $ax^2 + bx + c = a(x - r)(x - s), if\ ax^2 + bx + c = 0\ and\ r, s\ are\ solutions$.
3. $r + s = -\frac{b}{a}\ and\ r \times s = \frac{c}{a}\ for\ ax^2 + bx + c = 0\ where\ r\ and\ s\ are\ solutions.$

To illustrate these rules, take for example the following quadratic equation:

$$p^2 - 10p + 21 = 0$$

First we illustrate rule 1, the quadratic formula. From the formula we find that a=1, b=-10 and c =21. Using the formula we find two solutions for X:

$$x = \frac{10 \pm \sqrt{100 - 4 \times 1 \times 21}}{2} = \frac{10 \pm \sqrt{16}}{2} = \frac{10 \pm 4}{2}$$
$$= 5 + 2\ or\ 5 - 2, therfore\ 7\ and\ 3\ are\ the\ solutions$$

From rule 3 we deduce that $r + s = -\frac{-10}{1}\ and\ r \times s = \frac{21}{1}$ both have to hold, so r and s should be 7 and 3, as $7 + 3 = 10, 7x3 = 21$. From rule two we can see that the equation can indeed be written as $1(p - 3)(p - 7) = 0$, where 3 and 7 are the solutions.

Linear Equations with two unknowns

An example of a system of two equations with two unknowns is: $4x - 3y = 7$ and $2x + 5y = 23$. We need to solve for both x and y. We review two methods.

1. Substitution Method: First find the value of one variable in terms of the other, then substitute the value into the second equation.

Start by using the first equation to isolate x: $4x - 3y = 7 => 4x = 7 + 3y, x = \frac{7+3y}{4}$

Then substitute the value of x in terms of y in the second equation $2, 2(\frac{7+3y}{4}) + 5y = 23$

Algebra gives : $\frac{7+3+10y}{2} = 23$ → $7 + 13y = 46$ → $y = 3$ and $x = 4$

2. Elimination Method: We multiply both equations with a constant such that when equation 1 is subtracted from equation 2, one variable is removed. Thus leaving a simple linear equation. Take the same example as above.

Multiply equation 1 with 1 → $4x - 3y = 7$

Multiply equation 2 with 2 → $4x + 10y = 46$

And substract 2 from 1 to find $0x - 13y = -39$

Therefore, $y = 3$ and use this result to substitute 3 for y in equation 1 or 2 to get $x = 4$.

Final note

There is one very important fact one should remember whenever solving equations: When you multiply two or more factors, their product can only be zero if at least one of their factors is zero.

For example: $x(x + 3)(x - 2) = 0$
Whenever an equation is set up this way, the first solution that has to cross your mind is that $x = 0$. It is clearly not the only one, and with the same reasoning you can find that $x = -3$ and $x = 2$ are the other solutions (both creating factors of 0).

Chapter 3 Basic Notations

Summation

The abbreviated notion for summation is the following: $P_a + P_{a+1} + P_{a+2} \dots \dots \dots P_n = \sum_{k=a}^{n} P_a$ in which $k = a$ indicates the starting number of the sequence, and n indicated the number of repetitions.

An example of an economic application of such a summation is when calculating Price Indices (Inflation) for a group of goods:

$$\frac{Total\ cost\ of\ goods\ in\ final\ year}{Total\ cost\ of\ goods\ in\ Initial\ year} = \frac{\sum_{k=1}^{n} P_F^{(k)} Q^{(k)}}{\sum_{k=1}^{n} P_I^{(k)} Q^{(k)}} \times 100 = \text{Price Index}$$

Where, $Q^{(k)}$ is the amount of good K, $P_I^{(k)}$ is the price of good K in the first year under consideration, and $P_F^{(k)}$ is the price of good K in the final year under consideration.

We can distinguish between two kinds of price indices:

- Laspeyres Price Index: When the quantity consumed is based on the initial year for the above formula.
- Paashe Price Index: When the quantity consumed is based on the final year f or the above formula.

Properties of summation

When the variable has a subscript it means that there are different values available for this variable. When there is no subscript, it should be treated as a constant, not varying over time.

1. $\sum_{k=1}^{n} (s_k + r_k) = \sum_{k=1}^{n} s_k + \sum_{k=1}^{n} r_k$ (the additivity property)
2. $r \sum_{k=1}^{n} s_k = \sum_{k=1}^{n} r \times s_k$ (the homogeneity property)
3. $\sum_{k=1}^{n} (s_k + r \ ) = \sum_{k=1}^{n} s_k + nr$
4. $\sum_{k=1}^{n} K^2 = \frac{1}{6} n(n+1)(2n+1)$
5. $\sum_{k=1}^{n} K^3 = \left[\sum_{k=1}^{n} K \ \right]^2$
6. $(x+a)^n = \sum_{k=0}^{n} \binom{n}{k} x^k a^{n-k}$, where $\binom{n}{k} = \frac{n(n-1)(n-2)\dots\dots(n-m+1)}{m!}$

   Example: $\binom{7}{3} = \frac{7\times6\times5}{3\times2\times1} = 7 \times 5 = 35$
7. $\sum_{k=1}^{n} s_{k1} + \sum_{k=1}^{n} s_{k2} + \dots . + \sum_{k=1}^{n} s_{kb} = \sum_{r=1}^{n} [\sum_{k=1}^{n} s_{kr}]$

$$\text{Example: } \sum_{r=1}^{2}\left[\sum_{k=1}^{3}(2r+4k)\right] = \sum_{r=1}^{2}[(2r+4) + (2r+8) + (2r+12)]$$

$$= \sum_{r=1}^{2}[6r+24] = (6+24) + (12+24) = 66$$

Basics of Logic

A proposition, or statement, is an assertion that is either true or false.

1. Implication arrows are used to keep track in a chain of logical reasoning. The implication arrow $\Rightarrow$ points in the direction of the logical implication. $P \Rightarrow Q$ means that whenever proposition P is true, proposition Q is necessarily true as well.

2. When the logical implication is true in the other direction as well, we can write $P \Leftrightarrow Q$, which is a single logical equivalence. This arrow is called an equivalence arrow.

Q is a necessary condition for P ($P \Rightarrow Q$): a necessary condition for x to be a square (P) is that x be a rectangle (Q). Or as another example, 'the brakes are working (Q)' is a necessary condition for 'the bicycle is in good condition (P)'.

P is a sufficient condition for Q ($P \Rightarrow Q$): a sufficient condition for x to be a rectangle (Q) is that x be a square (P). A false example would be 'the brakes are working (P)' is a sufficient condition for 'the bicycle is in good condition (Q)', since the brakes could be working while the wheels could be missing. Hence P is not a sufficient condition for Q.

Mathematical proof

This book usually omits formal proofs of theorems, which are the most important results in mathematics. The emphasis lies on an intuitive understanding. Nevertheless:

Every mathematical theorem can be formulated as an implication $P \Rightarrow Q$, where P represents what we know (premises) and Q represents the conclusions (a series of propositions).

- A direct proof starts from the premises and works towards the conclusions.
- An indirect proof begins by supposing that the conclusion is not true, and reasons that on that basis the premises cannot be true either.

These proofs are both valid, based on the following equivalence:

$P \Rightarrow Q \quad \textit{is equivalent to} \quad \textit{not } Q \Rightarrow \textit{not } P$

Intuitively: 'if it is raining, the grass is getting wet' is the same thing as 'If the grass is not getting wet, then it is not raining'.

These two methods of proof both belong to *deductive reasoning*, reasoning based on consistent rules of logic. Many sciences use *inductive reasoning*, drawing general conclusions on the basis of a few (or many) observations. In mathematics inductive reasoning is not recognized as a form of proof.

Set Theory

In mathematics a collection of objects viewed as a whole is called a *set* and the objects inside the set are called *elements* or *members*. The easiest way to represent a set is to list it between brackets as follows:

$$S = \{a, b, c\}.$$

Two sets are considered to be equal if each element of A is an element of B and vice versa. Then we write $A = B$.

Not every set can be identified by listing all its elements. Some sets contain an infinite number of members. Such sets are common especially in economics. Think for example of the *budget set* in consumer theory. Infinite sets, but also finite sets, can be specified in the following way:

$S = \{typical\ member : defining\ properties\}$

Some standards are convenient to indicate the relationship between a set and its members

- $x \in S$ means that x is an element of S
- $x \notin S$ means that x is not an element of S
- $\emptyset$ is the symbol for an empty set, in other words, a set without elements
- $A \subseteq B$ means that A is a subset of B, and this is true when every member of A is also a member of B. The two sets A and B are only equal when $A \subseteq B$ and

We can distinguish three possible operations with sets:

Notation	Name	This (new) set consists of…	
$A \cup B$	A union B	The elements that belong to at least one of the sets A and B	$A \cup B = \{x: x \in A \ or \ x \in$
$A \cap B$	A intersection B	The elements that belong to both A and B	$A \cap B = \{x: x \in A \ and \ x$
$A \setminus B$	A minus B	The elements that belong to A, but not to B	$A \setminus B = \{x: x \in A \ or \ x \notin$

If two sets have no elements in common, they are said to be *disjoint*. Sets A and B can only be disjoint when $A \cap B = \emptyset$.

A collection of sets is usually referred to as a family of sets. Each set in any family is a subset of the *universal set* Ω. Hence, the complement of set A, given that set A is a subset of Ω, is the set of element of Ω that are not an element of A. The complement of set A can be denoted by $A^c = \Omega \setminus A$, or $\complement A$. It is always very essential to specify which universal set Ω is used to calculate the complement.

It is often useful to illustrate operations with sets by drawing Venn diagrams. A set is represented by a region in a plane, drawn so that all the elements belonging to a certain set are contained within some closed region of the plane.

These Venn diagrams show that some formulas can easily be found, for example:

- $A \cap B = B \cap A$
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

You can find four illustrative Venn diagrams on page 72.

Mathematical induction

Proof by mathematical induction is the following procedure:

Suppose that $A(n)$ is a statement for all natural numbers n and that

a. $A(1)$ is true

b. for each natural number k, if the induction hypothesis $A(k)$ is true, then $A(k+1)$ is true

Then $A(n)$ is true for all natural numbers n.

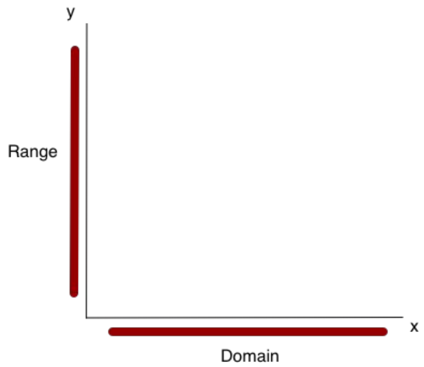
Chapter 4 Functions

Defining terms

One variable is a function of another if the first variable depends upon the second. Hence, a definite rule defines the relationship between the two variables. The independent variable is the variable that causes change in other variables. The dependent variable is the variable whose value depends on the value of other variables.

In the case of the relationship $y = f(x)$, x is the independent variable and y is the dependent variable. Hence the value of y will depend on the value of x.

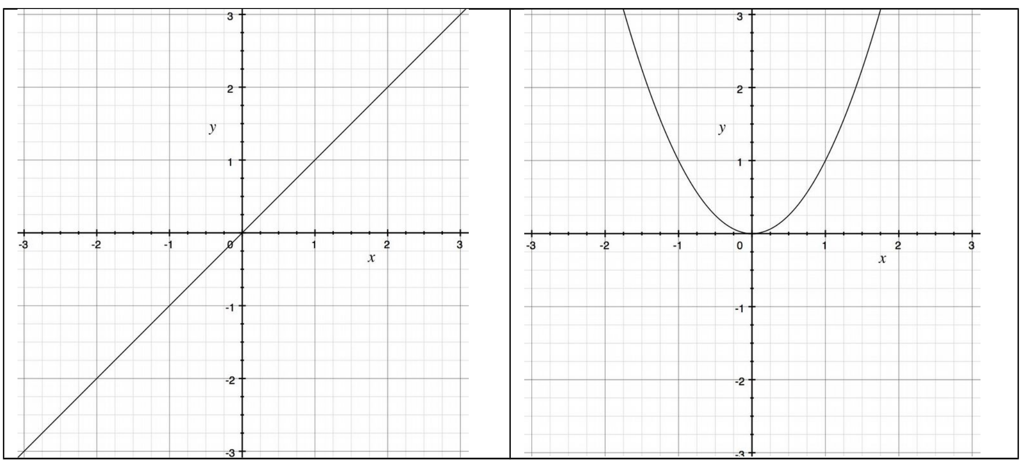
The *domain* of the function f as shown above is the set of all possible values for x (the independent variable). The *range* is the set of the corresponding values of y (the dependent value). To conclude, the range of a function depends on its domain.

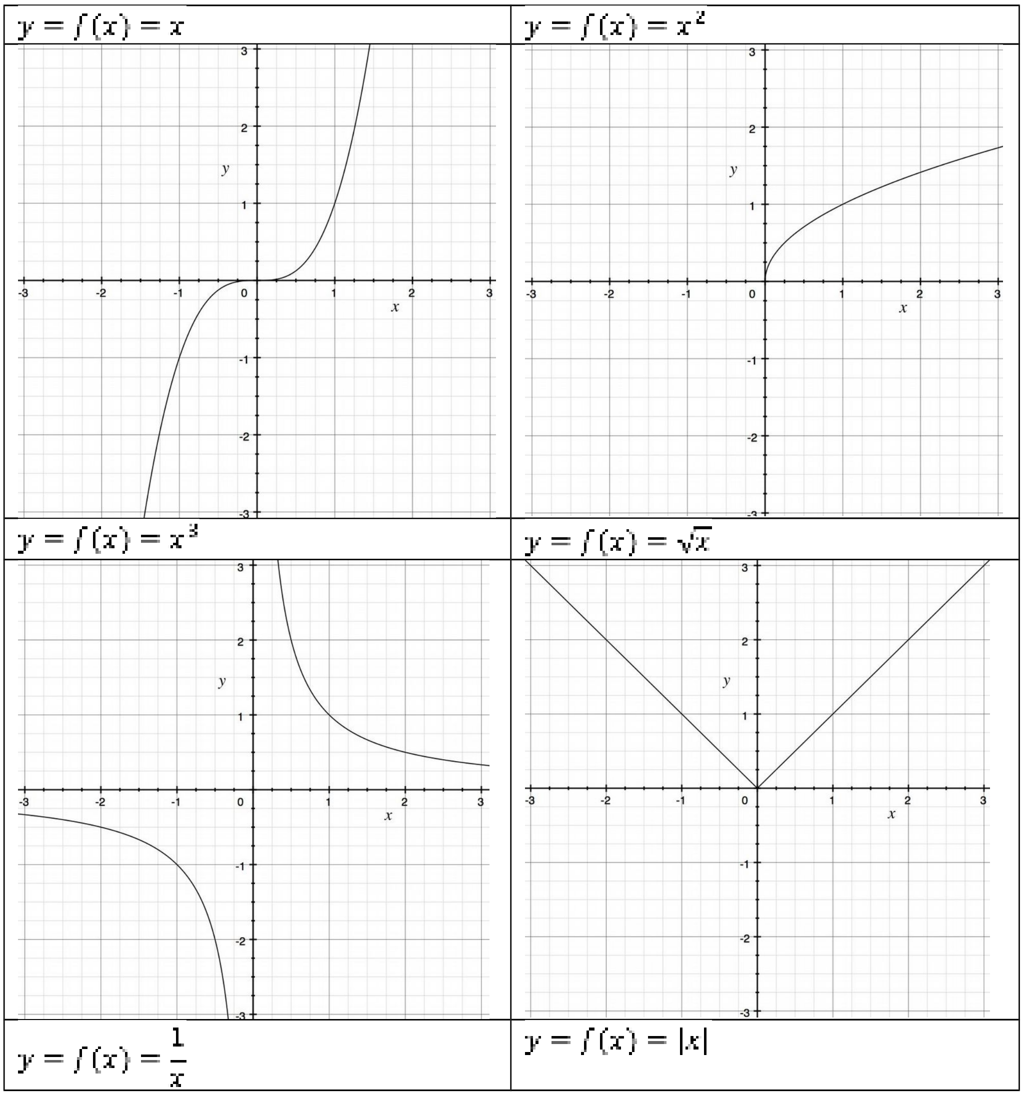


The $f(x)$ can simply be replaced by a different letter, $R(x)$ for example. However, $R(x)$ remains a function, and thus will also have the following property. The natural domain of the function f is the set of all real numbers and these real numbers must give only one y value for each x value.

Graphing functions

The xy plane as sketched above is also called the coordinate system. There are some elementary functions and matching graphs that you should know:





Linear functions

A linear function is written as $f(x) = ax + b$, where a and b are constant parameters. The properties of this function are:

- The graph is linear, a straight line
- a represents the slope
- b is the intercept on the Y-axis.

To find the equation of a straight line we can use 3 other methods:

1. *Point slope* formula**:** $y - y_1 = a(x - x_1)$**.** Use this formula when a point and a slope are known. For example: a line passes through the coordinates (2, 4) and its slope is 4. Find the equation of the line. The information given to us is: $(x_1, y_1) = (2,4)$ and $a = 4$. Thus, $y - 4 = 4(x - 2)$, $y = 4x - 4$.

2. Point-point formula: Use this formula when two points are given. For example, a line passes through (-3, -4) and (2,1). Find the equation of the line. To calculate the slope we can use the following formula: $a = \frac{y_2 - y_1}{x_2 - x_1}$. Note that x_1 cannot be equal to x_2 since the denominator would be equal to zero. Therefore, $a = \frac{1-(-4)}{2-(-3)}$, and $a = \frac{2}{3}$. Then we can proceed constructing the equation: $y - (-4) = \frac{2}{3}[x - (-3)]$, and $2 = \frac{2}{3}x - y$.

3. *Graphical approach*: Another method used with linear equation is graphing- you can graph, for instance, two equations and then find their intersection point.

Linear models and their applications in economics

We discuss two examples of applications of linear functions in economics.

1. The consumption function: $C = a + bY$. In this case, C represents consumption, b represents the marginal propensity to consume (for instance, when your income increases, how much are you going to spend of that increase?) and Y represents (national) income.

2. Supply and demand: basically you can have an equation for demand and an equation for supply (both equations will include a P for price). If you want to find the equilibrium price and the equilibrium quantity, you set the demand and supply equation equal to eachother, and then you simply solve for P to find the equilibrium price. By subsituting the equilibrium price back into one of the equations, you can find the equilbirum quantity as well. For example: take the following two equations, supply and demand respectively, $C = 100 + 20P,\ \ D = 80 + 40P$. Thus if you set both equations equal to eachother, you find that the equilibirum price equals 1. If you substitute 1 back into one of the equations, for instance $D = 80 + 40(1)$, you find that the equilibrium quantity will be 120.

Quadratic functions

The general quadratic equation is $f(x) = ax^2 + bx + c$, the graph of the equation is called a parabola. The parabola is always symmetric about the axis of symmetry, the minimum or maximum of the graph. This point is also called the *vertex* of the parabola.

To find this point we can use a shortcut:

- If $a > 0$, then $f(x) = ax^2 + bx + c$ has its *minimum* at $x = -\frac{b}{2a}$
- If $a < 0$, then $f(x) = ax^2 + bx + c$ has its *maximum* at $x = -\frac{b}{2a}$

Optimization using derivatives is explained in later, in chapter 6.

Polynomials

The next step is to consider *cubic functions* of the form $f(x)ax^3 + bx^2 + cx + d$. This is more complicated because the shape of the graphs depends strongly on the coefficients a,b,c and d. In fact, linear, quadratic and cubic functions are all examples of *polynomials.* A general polynomial of *degree n* is defined by:

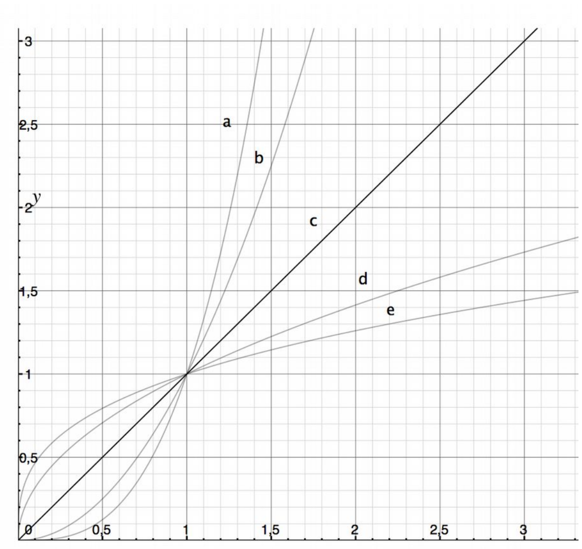
$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, where a's are constants and unequal to zero.

$$
\begin{array}{r|l|l}
x+2 & x^4 + 2x^3 + 2x^2 + 9 & x^3 + 2x - 4 \\
 & (-)x^4 (-)2x^3 & \\
\hline
 & \quad +2x^2 \quad +9 & \\
 & \quad (-)2x^2 (-)\ 4x + 0 & \\
\hline
 & \qquad -4x + 9 & \\
 & \qquad (-)4x\ (+)\ 8 & \\
\hline
 & \qquad\qquad 17 &
\end{array}
$$

In mathematics problems are often formulated such that a polynomial is equated to zero and to find all the possible solutions.
It is possible to divide polynomials in the following way. Take for example the division $(x^4 + 2x^3 + 2x^2 + 8) \div (x+2)$. Thus result has to be $x^3 + 2x - 4$, and the remainder is 17. Have a look at the calculation:
A rational function is a function $R(x) = P(x)/Q(x)$ that can be expressed as the ratio of two polynomials. It is only defined if $Q(x) \neq 0$.

Power functions

A power function is defined by *fx=Axr*, $x > 0$ and r and A are constants. x^r can be defined for all rational numbers r, that is for all fractional exponents. In the case of irrational numbers, we can take an approximation (for example of the number π) to make sure that x^r is defined. The shape of the graph also depends on the value of r. See the six planes above for an indication.



The five equations are respectively:

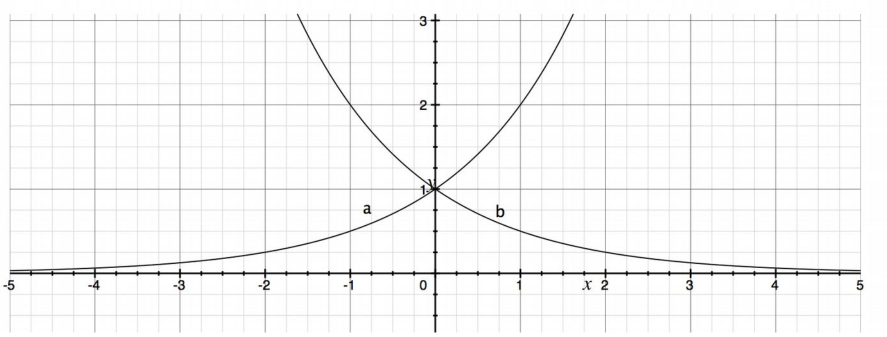
a. $f(x) = x^3$

b. $f(x) = x^2$

c. $f(x) = x^1$

d. $f(x) = x^{\frac{1}{2}}$

e. $f(x) = x^{\frac{1}{3}}$



Exponential functions

An exponential function is a function where the variable is the power: $f(x) = Aa^x$, where A and a are positive constants. The shape of the graphs depends on the value of a. See for example the two graphs below, in which a is $f(x) = 2^x$, and b is $fx=12x$. As you can see the functions are asymptotic, they do not reach zero or negative values.

Three examples of applications of exponential fuctions are the following:

1. Population Growth: $P(t) = population\ in\ base\ year \times (1 + \frac{rate\ of\ growth}{100})^{time\ in\ years}$
2. Compound Interest: A = $A = P\left[1 \pm \frac{r}{100}\right]^t$ where A = Total Amount, P = Initial amount, r = rate of change/interest rate (% per year in absolute terms), and t = Time in years.
3. Continuous Depreciation: When the value of the asset decreases with the same percentage each year then this process is called continuous depreciation, calculated in the following way:

   $$V(t) = P_i\left(1 - \frac{r}{100}\right)^t,$$

   $where\ V(t) is\ value\ of\ the\ assest\ on\ t\ year,$

   $P_i\ is\ the\ purchased\ price\ of\ the\ asset$

   $r\ is\ the\ rate\ of\ depreciation$

   $t\ is\ the\ time\ in\ years$

The *natural exponent* function is the function $f(x) = e^x$. The base of this function is the irrational number e and it has turned out to be the most important base for exponential functions.

Logarithmic functions

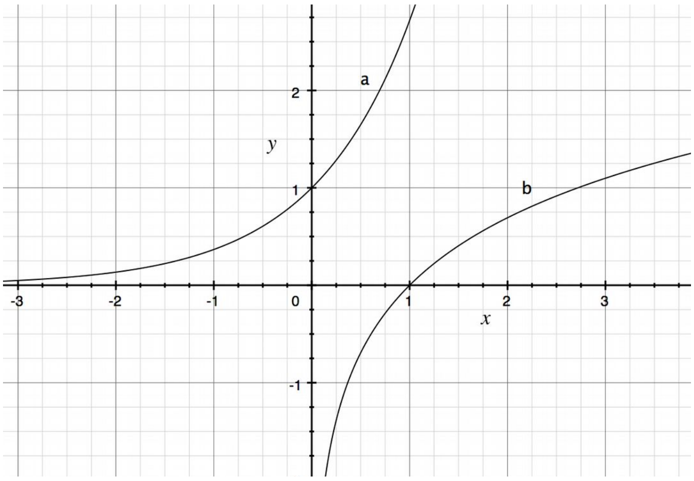
If $e^u = a$, then we call u the *natural logarithm* of a. The general form of a logarithmic function is therefore $e^{lnx} = x, for\ all\ positive\ values\ of\ x$.

There are some rules for working with the natural logarithm:

1. $\ln(xy) = lnx + lny$
2. $ln\frac{x}{y} = lnx - lny$
3. $lnx^p = plnx$
4. $ln1 = 0$
5. $lne = 1$
6. $x = e^{lnx}, x > 0$
7. $lne^x = x$

These rules also apply to logarithms with bases other than e. Any number a can be the base of a logarithm.

The graphs of the exponential function with base e, and the function of the natural logarithm are the following (respectively a and b):



Chapter 5 Functions continued

Shifting graphs

This graphs discussed how the graph of a function $y = f(x)$ can be transformed and therefore how it relates to the graphs of the functions $f(x) + d$, $f(x + d)$, $df(x)$, and $f(-x)$. d can be a positive or negative constant. For an illustration of the following cases, have a look at page 128.

1) $y = f(x) + d$: The graph moves upwards by d units if d is positive, and the graph will move down by d units if d is negative. The new graph is parallel to the first one.
2) $y = f(x + d)$: The graph moves d units to the left if d is positive, and the graph will move to the right by d units, if d is negative.
3) $y = df(x)$: The graph is stretched vertically if d is positive, and if d is negative, the graph will be stretched vertically and will be reflected about the x-axis.
4) $y = f(-x)$: The graph is reflected about the y-axis, as if the y-axis would be a mirror.

Introducing different types and properties of functions

The *sum* of two functions $f(x)$ and $g(x)$ is $F(x) = f(x) + g(x)$. $F(x) = f(x) - g(x)$ is called the *difference* between the two functions. A relevant example of a difference is the profit function, which is the difference between the revenue function and the cost function, $\pi(Q) = R(Q) - C(Q)$.

The *product* of two functions is $h(x) = f(x) \cdot g(x)$. And the *quotient* is $h(x) = \frac{f(x)}{g(x)}$.

A *composite* function is $y = f(g(x))$. $g(x)$ is called the *kernel* or *interior function*, while $f(\dots)$ is called the *exterior* function. First g applies on x, and on the result applies f.

An example of a composite function:
$f(x) = x^2 - 4, and\ g(x) = x - 2, find\ \ f(g(x))$.
$f(g(x)) = (x-2)^2 - 4$
$f\big(g(x)\big) = x^2 - 4x$

When a graph is symmetric about the y-axis, then $f(x)$ is called an *even* function. When the graph is symmetric about the origin, then $f(x)$ is called an *odd* function. A graph can also be symmetric about the line $x = a$, then $f(x)$ is said to be *symmetric about* a.

Inverse functions

An inverse function is a function that is the reverse of the given function, so if $f(x) = y\ then\ f^{-1}(y) = x$.

For example: The demand for crisps is described by the following function: $D(p) = 20 + 30p$.

However, as a producer of the crisps I do not want to know the demand for a specific price, but I want to decide on a certain output, and see what the price of that output will be. Therefore, I want to find the inverse of the demand function.

1. Rewrite the function as an equation: $D = 20 + 30p$.
2. Solve for p: $p = \frac{D-20}{30}$.
3. Switch p and D back and you have the inverse: $D^{-1}(p) = \frac{p-20}{30}$.

More formally, if and only if f is one-to-one, it has an inverse function g with domain B and range A. The function of g is given by the following rule: for each y in B, the value $g(y)$ is the unique number x in A such that $f(x) = y$. Then,

$$g(y) = x \iff y = f(x)$$
$$(x \in A, y \in B)$$

One-to-one means that the equation passes the vertical line test, meaning that, when drawing a vertical line on a certain value on the x-axis, the line can cross the function only once. In other words, for every x exists just one y. Vice versa is not true. If the graph does not pass the vertical line test, then the graph does not represent a function.

The domain of the inverse is the range of the original function, and the range of the inverse is the domain of the original function.

When two functions are inverses of each other, then the graphs of the two are symmetric about the line $y = x$.

General functions

Generally speaking, a function is a rule which to each element in a set A associates one and only one element in a set B. So if we denote the function by f, then the set A is called the domain of f, and set B is called the target or codomain of f. The definition of a function therefore needs to specify the domain, the target and the rule. Sometimes the words transformation and map, or mapping, are used for the same concept as a function.

The particular value of $f(x)$ is called the *image* of the element x. the *range* is the set of elements in B that are images of at least one element in A. The range is therefore a subset of the target.

Chapter 6 Derivatives

Slopes

When studying graphs we are usually interested in the slope of the graph. The steepness of a curve at a particular point can be defined as the slope of the tangent (a straight line that touches the curve at a certain point) to the curve at that specific point. The slope of the tangent to a curve at a particular point is the called the *derivative* of $f(x)$.

The definition of the derivative is the function f at point a, denoted by $f'(x)$, and is given by the formula:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

The equation for the tangent to the graph of $y = f(a)$ at the point $(a, f(a))$ is:

$$y - f(a) = f'(a)(x - a)$$

Finding the derivative

To find the value of the tangent you need to find the derivative and then substitute the given point a. The standard recipe for calculating the derivative $f'(a)$ is the following:

a) Add h to a and compute $f(a+h)$.
b) Compute the corresponding change in the function value: $f(a+h) - f(a)$
c) For $h \neq 0$, form the Newton quotient $\frac{f(a+h)-f(a)}{h}$.
d) Simplify the fraction and cancel h from the numerator and denominator as much as possible.
e) And then you find $f'(a)$ as the limit of this fraction as h tends to 0.

For example, to find $f'(x)$ when $f(x) = 4x^2$ by using this definition (the Newton quotient) of derivatives, you need to follow this procedure:

$$\frac{4(x+h)^2 - 4(x)^2}{h} = \frac{8hx + 4h^2}{h} = 8x + 4h$$

Since h approaches 0 (thus $4h$ will approach 0), the derivative will simply be $f'(x) = 8x$.

There are different ways of writing the derivative. The Leibniz notation, also called the *differential notation* is the following:

$$\frac{\mathrm{dy}}{\mathrm{dx}} \text{ or } \frac{\mathrm{df(x)}}{\mathrm{dx}} \text{ or } \frac{d}{dx} f(x)$$

Increasing or decreasing

The definitions of increasing and decreasing functions are the following:

- If $f(x_2) \geq f(x_1)$ whenever $x_2 > x_1$, then f is *increasing* in interval I.
- If $f(x_2) > f(x_1)$ whenever $x_2 > x_1$, then f is *strictly increasing* in interval I.
- If $f(x_2) \leq f(x_1)$ whenever $x_2 > x_1$, then f is *decreasing* in interval I.
- If $f(x_2) < f(x_1)$ whenever $x_2 > x_1$, then f is *strictly decreasing* in interval I.

When we relate this definition of increasing and decreasing to the derivative, we have the following propositions:

- $f'(x) \geq 0$ for all x in the interval I $\Leftrightarrow$ f is increasing in I
- $f'(x) \leq 0$ for all x in the interval I $\Leftrightarrow$ f is decreasing in I
- $f'(x) = 0$ for all x in the interval I $\Leftrightarrow$ f is constant in I

This makes sense since the derivative represents the slope of a function, and thus if a slope is for instance 2, the line will rise from left to right: the function is increasing.

For examine, consider the function $f(x) = 2x^2 - 2$ and define where the function is increasing or decreasing.

1. Find the derivative: $f'(x) = 4x$.
2. Set the derivative equal to zero, and solve for x. Hence, $x = 0$.
3. Substitute a value lower than 0 (for instance -2) into the first derivate, and a value higher than 0 (for instance 2). This will allow you to find the intervals for which the function will be increasing or decreasing:

$f'(x) = 4(-2)$	$f'(x) = 4(2)$
-8 = decreasing	8= increasing

The conclusion is that $f(x)$ is increasing when $x > 0$ and decreasing when $x < 0$.

Derivatives as rates of change

A derivative measures change, to be precise the average rate of change of f over the interval from a to a+h. The slope of the tangent line at a particular point is the *instantaneous rate of change* and is therefore $f'(a)$. The *relative rate of change* of f at point a is $\frac{f'(a)}{f(a)}$. This measure can be used to describe, for instance, how much a variable changed this year (written as a percentage). The relative rate of change is sometimes called the *proportionate rate of change*.

In economics the word *marginal* is widely used to indicate a derivative. For example the marginal propensity to consume and the marginal product are both measures of change.

Limits

Writing $\lim_{x\to a} f(x) = A$ means that we can make $f(x)$ as close to A as we want for all x sufficiently close to (but not equal to) a. Or in other words we can say that $f(x)$ has the number A as its limit, as x tends t a.

There are some important rules for limits. If $\lim_{x\to c} f(x) = C$ and $\lim_{x\to d} g(x) = D$, then:

1) $\lim_{x\to c}\big(f(x) \pm g(x)\big) = C \pm D$
2) $\lim_{x\to c}\big(f(x) \cdot g(x)\big) = C \cdot D$
3) $\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{C}{D}$, if $D \neq 0$
4) $\lim_{x\to c}(f(x))^r = A^r$ (if A^r is defined and r is any real number)

Differentiation

Differentiation is a process to find the derivative. If a limit exists, then a function is differentiable at x. If the function is a constant, then its derivative is 0. Also additive constants disappear in the process. Multiplicative constants are however preserved.

The basis of differentiation is the *power rule:*

$$f(x) = x^a \to f'(x) = ax^{a-1}$$

In case of the *sum or difference* of two functions, differentiation is still possible:

$$F(x) = f(x) \pm g(x) \to F'(x) = f'(x) \pm g'(x)$$

However, when we have to differentiate a product of two functions, we need to use the *product rule:*

$$F(x) = f(x) \cdot g(x) \to F'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

To illustrate this product rule, consider this example. Find the derivative of $F(x) = (x^2 - 1) \cdot (x + 4)$. The process is the following:

1. $f(x) = (x^2 - 1)$ $\quad f'(x) = 2x$, by using the power rule.
2. $g(x) = (x + 4)$ $\quad g'(x) = x$, by using the power rule.
3. $F'(x) = (2x)(x + 4) + (x^2 - 1)(x)$, by using the product rule.
4. conclusion: $F'(x) = x^3 + 2x^2 + 7x$

In case of a division of two functions, we need to use the *quotient rule:*

$$F'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g^2}$$

Again, we illustrate this rule by an example. Find the derivative of $F(x) = \frac{2x^2+1}{4x-2}$

1. $f(x) = 2x^2 + 1$, $f'(x) = 4x$
2. $g(x) = 4x - 2$, $g'(x) = 4$
3. Conclusion: $F'(x) = \frac{4x \cdot (4x-2) - (2x^2+1) \cdot 4}{(4x-2)^2}$

In case of a composite function, hence $f(u(x))$, we need to use the *chain rule*

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

This implies that if y is a differentiable function of u, and u is a differentiable function of x, then y is a differentiable function of x. When $y = u^a$ then $y' = au^{a-1}u'$. U can be considered as a separate function dependent on x.

Formulated formally, if g is differentiable at x_0 and f is differentiable at $u_0 = g(x_0)$, then $F(x) = f(g(x))$ is differentiable at x_0, and

$$F'(x_0) = f'(u_0)g'(x_0) = f'(g(x_0))g'(x_0)$$

So to differentiate a composite function, first differentiate the exterior function w.r.t. the kernel, and then multiply by the derivative of the kernel.

Higher order derivatives

$f'(x) = first\ derivative$
$f''(x) = second\ derivative$

The procedure for finding the second derivative is based on the same properties as explained above, w.r.t. the first derivative. For example, find $f'(x)$ and $f''(x)$ for $f(x) = 4x^3 - 2x + 1$. The answers are $f'(x) = 12x^2 - 2$ and $f''(x) = 24x$.

We can find the second derivative, but we can also find for instance the tenth derivative; the n^{th} derivative is written in the following way: $y^n = f^n(x)$.

Convexity and concavity

1) $f\ is\ convex\ on\ the\ interval\ if\ f''(x) \geq 0\ for\ all\ x's\ on\ the\ interval$
2) $f\ is\ concave\ on\ the\ interval\ if\ f''(x) \leq 0 for\ all\ x's\ on\ the\ interval$

The difference between convex and concave will be important when using economic models, but we will discuss this topic later in detail.

Exponential Function

The exponential function is a particular function with particular properties. It is differentiable, strictly increasing and convex.

The following properties hold for all exponents s and t:

a) $e^s e^t = e^{s+t}$

b) $\frac{e^s}{e^t} = e^{s-t}$

c) $(e^s)^t = e^{st}$

Moreover:

a) $e^x \longrightarrow 0$ as $x \longrightarrow -\infty$

b) $e^x \longrightarrow \infty$ as $x \longrightarrow \infty$

To differentiate an exponential function use the following rule:

$$f(x) = e^x \rightarrow f'(x) = e^x$$

However, if you have to find the derivative of $f(x) = e^{x^2}$, you have to use the chain rule:

$$f(x) = e^u \rightarrow f'(x) = u' \cdot e^u$$

To differentiate other exponential functions we have to start from the following formula: $a^x = (e^{\ln a})^x = e^{(\ln a)x}$

Therefore when $y = a^x$ then $y' = a^x \ln a$

Logarithmic Functions

The natural logarithmic function $g(x) = \ln x$ is differentiable, strictly increasing and concave in (0,∞).

By definition $e^{\ln x} = x$ for all $x > 0$, and $\ln e^x = x$ for all x. The following properties hold for all positive x and y.

a) $\ln(xy) = \ln x + \ln y$

b) $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$

c) $\ln(x^p) = p \ln x$

To differentiate a natural logarithmic function, use the following formula:

$$f(x) = lnx \rightarrow f'(x) = \frac{1}{x}$$

For example, the derivative of $f(x) = lnx + x^2$ is $f'(x) = \frac{1}{x} + 2x$.

Chapter 7 Application of Derivatives:

Implicit Differentiation

1. Implicit Differentiation is used when the function is not in the form a= f(b), but the variables x and y could be on either side of the equation.
2. Instead of getting the equations to the form a=f(b), just take derivative on both sides (REMEMBER: re-write 'a' as f(b) considering 'a' to be a function of 'b')
3. Solve the equation for y'

For example, consider the equation $2a^2 + 4ab = 2b^2$. To be able to differentiate, consider it as $2a^2 + 4af(b) = 2[f(a)]^2$. Remember that we have to use the chain and product rule where applicable:

$$2 \times 2 \times a + 4a^0 f(a) + 4af'(a) = 2 \times 2 \times f(a) \times f'(a)$$
$$4a + 4f(a) + 4af'(a) = 4f(a)f'(a)$$
$$4a + 4f(a) = f'(a)[4f(a) - 4a]$$
$$f'(a) = \frac{4a + 4f(a)}{4f(a) - 4a} = \frac{4a + 4b}{4b - 4a} = b'$$

In economics implicit differentiation is widely used, because many economic problems are formulated in a system of implicit equations relating different variables.

Inverse and Differentiation

If a function is differentiable, and it is strictly increasing or decreasing then for that interval the function will have an inverse function (b in the formulation below). Formally:

$$for\ a\ function\ b = f(a), \qquad h'(b_i) = \frac{1}{f'(a_i)}$$

For example, if we want to find the inverse of the function $f(x) = e^x$, then we need to follow this reasoning: $y = e^x = f(x)$ and therefore the inverse is $x = g(y) = lny$.To find the derivative we use this rule: $g'(y) = \frac{1}{f'(x)}\ from\ the\ equation\ given\ above$. And the result is:

$$\frac{1}{y} = \frac{1}{e^x} \qquad therefore, the\ inverse\ function\ is\ e^x = y$$

Understanding Linear Approximations

When there is a complex function then we can sometimes replace it with a linear function with a similar graph. This is called a linear approximation to function $f(x)$ about a certain point $x = a$. The formula for the approximation is:

$f(x) \approx f(a) + f'(a)(x - a)$, when x is close to a.

For example, to find the approximate value of $(1.003)^{60}$, consider this equation to be $f(x) = x^{60}, where\ x = 1.003,\ Now\ let\ us\ assume\ x = 1.003 \approx a = 1$.

By using the formula we find $f(1.003^{60}) \approx f(1) + f'(1^{60})(1.003 - 1)$. Then we solve the part $f'(1^{60}) = 60 \times 1^{59}$ = 60. And therefore, $f(1.003^{60}) \approx 1 + 60(0.003) = 1.018$.

In general terms, when we consider a differentiable function $f(x)$, then the expression $f'(x)dx$ is called the *differential* of $y = f(x)$, and its denoted by dy. So, the formal expression is $dy = f'(x)dx$. Here dx is a single symbol representing the change in the value of x and it is not a multiplication of d and x.

The following rules hold:

1. $d(uf + vg) = udf + vdg$

$(where\ d\ stands\ for\ differentiate, \quad u, v\ are\ constants, \quad f, g\ are\ functions)$

2. $d(fg) = gdf + fdg$
3. $d\left(\frac{f}{g}\right) = \frac{gdf - fdg}{g^2}$

Understanding Polynomial Approximations

When there is a complex polynomial then we can sometime replace it by an approximation to make calculations easier. For example a quadratic approximation:

For a fixed value $a = u$

$$f(a) \approx f(u) + f'(u)(a-u) + \frac{1}{2} f''(u)(a-u)^2$$

Other approximations of polynomials with a higher order follow the same logic:

$$f(a) \approx f(u) + \frac{f'(u)}{1!}(a-u) + \frac{f''(u)}{2!}(a-u)^{\wedge}2 + \ldots\ldots\ldots + \frac{f^n(u)}{n!}(a-u)^n$$

Taylor's formula

One of the main results of mathematical analysis in economics is called Taylor's formula and is a remedy for the problem of the difference between the real formula and an approximation. The difference between the two is called the *remainder*. Therefore, by definition:

f,x.≈f,0.+,1-1!.,f-′.,0.x+,1-2!.,f-′′.,0.,x-2.+…+,1-n!.,f-,n..,0.,x-n.+,R-n+1.(x)

The explicit formula for the remainder R is given by the following proposition, called the Lagrange form:

$$R_{n+1}(x) = \frac{1}{(n+1)!} f^{(n+1)}(c) x^{n+1}$$

Based on the supposition that a function is n+1 times differentiable in an interval excluding x and 0 t (the number c is also contained in this interval).

Using this formula we get to *Taylor's formula:*

$$f(x) \approx f(0) + \frac{1}{1!}f'(0)x + \cdots + \frac{1}{n!}f^{(n)}(0)x^n + \frac{1}{(n+1)!}f^{(n+1)}(c)x^{n+1}$$

Elasticities

The *price elasticity of demand* measures by what percentage the quantity demanded changes when the price increases by 1%. The symbol used for price elasticity is $EL_p\, D(p)$. The price elasticity with respect to demand can be calculated using the formula:

$$EL_p\, D(p) = \frac{p}{D(p)} \times \frac{dD(p)}{dp}$$

The general formulation of the elasticity of a function with respect to the variable x is:

$$EL_x f(x) = \frac{x}{f(x)} \times f'(x)$$

For example, find the price elasticity using the $D(p) = \frac{p-1}{p+1}$ by the following procedure:

1. $EL_p D(p) = \frac{p}{D(p)} \times D'(p)$ is the general formulation of the solution.
2. $EL_p D(p) = \frac{p\,(p+1)}{(p-1)} \times \frac{(p+1)-(p-1)}{(p+1)^2}$. $D'(p)$ can be found using the quotient rule.
3. $EL_p D(p) = \frac{p}{(p-1)} \times \frac{2}{p+1}$ is given by rewriting and therefore, the price elasticity is $EL_p D(p) = \frac{2p}{(p-1)^2}$.

Continuity

Graphically speaking, a function is said to be continuous if its graph is connected, that is, it has no breaks. Otherwise a function is called discontinuous. In terms of limits, a function is continuous at point $x = a$ if:

$$\lim_{x \to a} f(x) = f(a)$$

Therefore, three conditions need to be fulfilled, otherwise a function is called discontinuous:

i. The function must be defined at $x = a$.
ii. The limit of $f(x)$ as x tends to a needs to exist.
iii. This limit must be exactly equal to $f(a)$.

The four properties of continuous functions are the following. If f and g are continuous at a, then:

1. $f+g$ and $f-g$ are continuous at a
2. fg and $\frac{f}{g}$ are continuous at a
3. $[f(x)]^r$ is continuous at a if $[f(a)]^r$ is defined
4. If f is continuous and has an inverse on the interval I, then its inverse f^{-1} is continuous on the same interval.

Any function that can be constructed from continuous functions by addition, subtraction, multiplication, division and composition is also continuous at all points where it is defined.

More on limits

We say that a limit exists if it tends towards a number a, sufficiently close. If a limit does not exist, we say that it tends to infinity. In that case a *vertical asymptote* exists.

In some cases the value of a limit depends on from which side you start reasoning. The notation of one sided limits, respectively from below or from above, is as follows:

$\lim_{x \to a^-} f(x) = B \text{ or } f(x) \to B \text{ as } x \to a^-$
and

$\lim_{x \to a^+} f(x) = A \text{ or } f(x) \to A \text{ as } x \to a^+$

Now we can define one sided continuity. In the first case we say that the function is *left continuous* and in the second case we say that the function is *right continuous*.

We can also define the limit of a function when x tends to infinity. Then encounter horizontal asymptotes, and we write:

$\lim_{x \to \infty} f(x) = A \text{ or } f(x) \to A \text{ as } x \to \infty$

When studying composites of functions with respect to continuity we have to consider the following properties, especially the last two:

If $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to a$, then:

- $f(x) + g(x) \to \infty$
- $f(x)g(x) \to \infty$
- $f(x) - g(x) \to ?$
- $f(x)/g(x) \to ?$

The relationship between continuity and differentiability is relatively simple. If a function is differentiable at $x = a$ then the function is continuous at $x = a$. However, if there is a kink in the graph (and function) at point a, then the function is continuous but not differentiable at that point.

The conclusion of this discussion about continuity of functions is that, if a function is continuous in a closed interval $[a, b]$, then according to the *intermediate value theorem*:

i. If $f(a)$ and $f(b)$ have different signs, then there is at least one c in the interval such that $f(c) = 0$.
ii. If $f(a) \neq f(b)$, then for every intermediate value y in the open interval between $f(a)$ and $f(b)$ there is at least one c such that $f(c) = y$.

The intermediate value theorem is normally used to state that an equation $f(x) = 0$ has a solution in a given interval. *Newton's method* also approximates the location of the zero. The method generates a sequence of points that converges to a zero quickly, given by the formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Formally we say that a sequence is converging to a number s if s_n can be made arbitrarily close to s by choosing n sufficiently large: $\lim_{n\to\infty} s_n = s$. In the opposite case a sequence is said to diverge.

Chapter 8 Optimal Points

Extreme points

The extreme points of a function are where it reaches its largest and its smallest values, the maximum and minimum points. Formally,

d) $c \in D$ is the maximum point for $f \Leftrightarrow f(x) \leq f(c)$ for all $x \in D$
e) $d \in D$ is the minimum point for $f \Leftrightarrow f(x) \geq f(d)$ for all $x \in D$

Where the derivative of the function equals zero, $f'(x) = 0$, the point x is called a *stationary point* or *critical point*. For some point to be the maximum or minimum of a function, it has to be such a stationary point. This is called the *first-order condition*. It is a necessary condition for a differentiable function to have a maximum of minimum at a point in its domain.

Stationary points can be local or global maxima or minima, or an inflection point. We can find the nature of stationary points by using the first derivative. The following logic should hold:

8. If $f'(x) \geq 0$ for $x \leq c$ and $f'(x) \leq 0$ for $x \geq c$, then $x = c$ is the maximum point for f.
9. If $f'(x) \leq 0$ for $x \leq c$ and $f'(x) \geq 0$ for $x \geq c$, then $x = c$ is the minimum point for f.

Is a function concave on a certain interval I, then the stationary point in this interval is a maximum point for the function. When it is a convex function, the stationary point is a minimum.

Economic Applications

Take the following example, when the price of a product is *p*, the revenue can be found by $R = 5p^2 + 20p + 16$. What price maximizes the revenue?

5) Find the first derivative: $R' = -10p + 20$
6) Set the first derivative equal to zero, $0 = -10p + 20$, and solve for p.
7) The result is $2 = p$.

Before we can conclude whether revenue is maximized at 2, we need to check whether 2 is indeed a maximum point. The easiest way is to find the second derivative, and if the second derivative is negative for 2, then 2 is indeed a maximum point. The second derivative will be positive if 2 is a minimum point. The second derivative is $R'' = -10$. Thus, 2 is indeed the maximum point.

Extreme Value Theorem

The reasoning above is mainly based on the extreme value theorem: Suppose that f is a continuous function in a closed and bounded interval $[a, b]$. Then there exists a point d in

$[a, b]$ where f has a minimum, and a point c in $[a, b]$ where f has a maximum, so that:

$$f(d) \leq f(x) \leq f(c), \text{ for all } x \text{ in } [a, b]$$

Every extreme must be one out of three options:

8. Interior points where $f'(x) = 0$
9. End points of the closed and bounded interval
10. Interior points where f' does not exist

Another example: Find the maximum and minimum values of the function $f(x) = 2x^2 - 4x + 5$ for the interval $x \in [0,5]$.

1. Find the stationary points using the first order condition:
   2. $f'(x) = 4x - 4$, $0 = 4x - 4$, $x = 1$
   3. $f(1) = 2(1)^2 - 4(1) + 5$, $f(1) = 3$
4. Find the y-values of the end points of the interval:
   5. $f(0) = 2(0)^2 - 4(0) + 5$, $f(0) = 5$
   6. $f(5) = 2(5)^2 - 4(5) + 5$, $f(5) = 35$
7. By looking at all three points (0,5) (1,3) (5,35) we can see that (1,3) is the minimum value.

Local maximums and minimums

Although usually the global maxima or minima are of interest for economists, also local extreme points are sometimes relevant. Formally the function f has a local, or relative, maximum at c if there exists an interval (α, β) about c such that $f(x) \leq f(c)$ for all x in (α, β) which are in the domain of f.

To find local extreme points the first-derivative has to be used as well. The following reasoning holds:

3. If $f'(x) \geq 0$ throughout some interval to the left of point c and $f'(x) \leq 0$ to the right of the point, then $x = c$ is a local maximum point for f.
4. If $f'(x) \leq 0$ throughout some interval to the left of point c and $f'(x) \geq 0$ to the right of the point, then $x = c$ is a local minimum point for f.
5. If $f'(x) > 0$ throughout some interval both to the left and to the right of point c then $x = c$ is not a local extreme point for f. The same holds for $f'(x) < 0$.

A more formal way to know if the stationary points are local extreme points, is by use of the second derivatives:

1. $f'(c) = 0$ and $f''(c) < 0$, then $x = c$ is a strict local maximum point
2. $f'(c) = 0$ and $f''(c) > 0$, then $x = c$ is a strict local minimum point
3. $f'(c) = 0$ and $f''(c) = 0$, then we do not know the nature of the point

Inflection Points

If in the last case the second derivative f'' changes sign at point c, then that point is called an inflection point. The meaning of an inflection point is that the function f is changing from concave to convex or vice versa. An example of a function with an inflection point is $f(x) = \sqrt[3]{x}$. A function is called concave if the line segment joining any two points on the graph is below or on the graph. For convex functions the reasoning is vice versa. Formally:

$$f''(x) < 0 \; for \; all \; x \in (c,d), f(x) is \; concave \; in \; (c,d)$$
$$f''(x) > 0 \; for \; all \; x \in (c,d), f(x) is \; convex \; in \; (c,d)$$

Chapter 9 Integrals

Indefinite integrals
To reverse the process of differentiation we use integration, or in other words we take the anti derivative. Formally the *indefinite integral* is

$$\int f(x)dx = F(x) + C$$

when, $F'(x) = f(x)$.

The symbol in front is called the integral sign and $f(x)$ is the *integrand*. To indicate that the variable of integration is x, it is written dx. It is called the indefinite integral because $F(x) + C$ should be seen as a class of functions, all having the same derivative $f(x)$.

By definition the derivative of an indefinite integral equals the integrand:

$$\frac{d}{dx}\int f(x)\,dx = f(x)$$

Thus, integration and differentiation cancel each other out.

Some important formulas directly result from differentiation formulas as stated in other chapters (a is a constant):

- $\int x^a\,dx = \frac{1}{a+1}x^{a+1} + C$
- $\int \frac{1}{x}dx = \ln|x| + C$
- $\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$
- $\int a^x\,dx = \frac{1}{\ln a}a^x + C$
- $\int af(x)\,dx = a\int f(x)\,dx$
- $\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$

We give an example of solving an integration problem. Consider the function $f = 4x^2$ then the derivative is $f'(x) = 8x$. If we take the integral of this derivative we get the indefinite integral: $\int 8x = 4x^2 + c$.

Thus, to integrate you first add a number to the exponent on the variable x and you divide the number in front of x by this new exponent, this rule can be written as: $\frac{1}{n+1}x^{n+1} + C$. Then we need to add an undefined constant, to cover for the possibility of a constant that has disappeared in the differentiation process. To illustrate this, consider the following example: If we need to differentiate the function $f(x) = x^2 + 4x + 2$, then the derivative would be $f'(x) = 2x + 4$. Integrating this function would give us $x^2 + 4x$. However, we have no way of knowing for sure if some constant was part of the original function, in this case 2. That is why we need to add c at the end of the indefinite integral; the c represents any number that could have been behind $x^2 + 4x$.

Definite Integrals

The exact area under a curve (between the graph and the x-axis) is given by the definite integral. The definite integral has a lower and a upper limit of integration. A definite integral is defined in the following way:

$$\int_a^b f(x)dx = |_a^b F(x) = F(b) - F(a), where\ F'(x) = f(x) for\ all\ x \in (a, b)$$

For example, to solve the integral $\int_2^4 (x^2 - 4x)dx$ proceed like this:
$\int_2^4 (x^2 - 4x)dx = |_2^4 \left(\frac{1}{3}x^3 - 2x^2\right) = [\frac{1}{3}(4)^3 - 2(4)^2] - [\frac{1}{3}(2)^3 - 2(2)^2] = \frac{16}{3} - \frac{16}{3}$ = 0.
Thus, there is no area under the curve.

If the defined area under $f(x)$ is negative, then the area you are finding is below the x-axis. However, note that you are still finding an area. There is simply a negative side in front of the integral. The area under the x-axis is simply subtracted from the total area.

Properties of Definite Integrals

1. $\int_a^b f(x)dx = -\int_b^a f(x)dx$
2. $\int_a^a f(x)dx = 0$
3. $\int_a^b \propto f(x)dx = \propto \int_a^b f(x)dx$, where α is an arbitrary number
4. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$
5. $\int_a^b [\alpha f(x) + \beta g(x)]dx = \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx$

Moreover, the derivative of the definite integral with respect to the upper limit of integration is equal to the integrand (the expression you are integrating) at that limit:

$$\frac{d}{dz}\int_a^z f(x)dx = F'(z) = f(z)$$

The derivative of a definite integral with respect to the lower limit of integration is equal to minus the integrand evaluated at the limit:

$$\frac{d}{dt}\int_{z}^{w} f(x)\,dx = -F'(z) = -f(z)$$

These two rules can be generalized into the following formula:

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x)dx = f\big(b(t)\big)b'(t) - f\big(a(t)\big)a'(t)$$

Economic application

An example of an application in economics of the integral is the calculation of the consumer and producer surplus. See page 313 for a graphical illustration. The two formulas are:

$$CS = \int_{0}^{Q*} [f(Q) - P^{*}]dQ$$

$$PS = \int_{0}^{Q*} [P^{*} - g(Q)]dQ$$

Integration by Parts

To integrate an equation by parts we need to first re-write the equation in the form of $f(x) \times h'(x)dx$. In words this is a function for the variable x multiplied by the derivative of another function in terms of this variable x. Just as the derivative of a product is not the product of the derivatives, we also need to use a different formula for integration, which is:

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

A similar formula exists for definite integrals:

$$\int_{a}^{b} f(x)g'(x)dx = |_{a}^{b} f(x)g(x) - \int_{a}^{b} f'(x)g(x)dx$$

For example, to solve the indefinite integral $\int 2x^4 dx$, we have to rewrite it to $\int (x)^3 \times 2x$. Then $f(x) = x^3 and\ h(x) = x^2, since\ 2x\ is\ the\ derivative\ of\ x^2$, implying that the formula is:

$$\int x^3 \times 2x(dx) = x^3 \times x^2 - \int 3x^2(dx) \times x^2$$

Therefore, $\int_q^p f(x)h'(x)dx = [_q^p f(x)h(x) - \int_q^p f'(x)dxh(x)$

Integration by Substitution

To integrate by substitution the equation should be re-written as a composite function $f\big(h(x)\big) \times h'(x)dx$. Then the rule for integration by substitution is:

$$\int f\big(g(x)\big)g'(x)dx = \int f(u)du, \text{ where } u = g(x)$$

For complicated integrals of the form $\int G(x)dx$ we can use the following procedure:

- Pick out a part of $G(x)$ and introduce it as a new variable: $u = g(x)$
- Compute $du = g'(x)\,dx$
- Using the substitution $u = g(x)$, $du = g'(x)$, transform $\int G(x)\,dx$ to $\int f(u)\,du$
- Then find $\int f(u)\,du = F(u) + C$
- Replace u by $g(x)$. This gives the final answer $\int G(x)\,dx = F\big(g(x)\big) + C$

Infinite Intervals

If a function f is continuous, then the integral $\int_a^b f(x)\,dx$ is defined for all $b \geq a$. If the limit $b \to \infty$ exists, then f is integrable over the infinite interval $[a, \infty)$ and the improper integral is said to converge:

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$

If the limit does not exist, then the integral is said to diverge. In the case of the interval $(-\infty, \infty)$, the improper integral is defined as:

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^0 f(x)\,dx + \int_0^\infty f(x)\,dx$$

If both integrals on the right hand side converge, then the entire integral is said to converge. Otherwise, it diverges.

An alternative test for convergence exists. Suppose that f and g are continuous and $|f(x)| \leq g(x)$ for all $x \geq a$. Then $\left|\int_a^\infty f(x)\,dx\right| \leq \int_a^\infty g(x)\,dx$, and if one converges than the other also converges.

Reviewing Differential Equations

Differential equations are equations where the unknowns are functions, and where the derivatives of these functions also appear. In the case of differential equations, the independent variable is usually denoted as t, because usually time is the independent variable.

To find all functions that solve $\dot{x}(t) = f(t)$ ($\dot{x}$ is the derivative of x with respect to time) we know that the general solution is an indefinite integral:

$$x(t) = \int f(t)\,dt + C$$

More complicated versions of differential equations are often related to growth rates. There are two general types of differential equations, separable and linear differential equations.

A separable differential equation is of the type $\dot{x} = f(t)g(x)$ and the unknown function is $x = x(t)$. To solve we have to take four steps.

- Write the equation differently: $\frac{dx}{dt} = f(t)g(x)$
- Separate the variables: $\frac{dx}{g(x)} = f(t)\,dt$
- Integrate each side: $\int \frac{dx}{g(x)} = \int f(t)\,dt$
- To find a solution for the equation in step 1, evaluate both integrals, and re-write for x if possible.

A first-order linear equation is of the form $\dot{x} + a(t)x = b(t)$ where $a(t)$ and $b(t)$ are continuous functions in a certain interval. $x = x(t)$ is the unknown in the function.

Chapter 10 Rate of Interest and Values

Interest

Interest rates are usually quotes as annual rates, also called nominal rates, even if the actual interest period is different. The interest period is the time between two successive dates when interest is added to the account. To get the periodic rate of interest you need to divide the nominal rate of interest by the number of periods. Therefore the principle increases as follows:

$$S(t) = S_0(1+r)^t$$

To be able to compare interest rates, usually the concept of the *effective yearly rate* and its independent of the value of the initial principle S:

$$R = \left(1+\frac{r}{n}\right)^n - 1$$

Continuous Compounding

When we assume continuous compounding, it implies that we expect interest to be compounded regularly and frequently. To show how much a principle S will have increased after t years with annual interest r, we use the formula:

$$S(t) = S_0 e^{rt}$$

When calculating the present value of a payment due in the future we have to consider the interest rate. If the interest rate is p& per year and $r = \frac{p}{100}$, an amount K that is payable in t years has the present value of:

1. $K(1+r)^{-t}$, in the case of annual interest payments, and
2. Ke^{-rt}, in the case of continuous compounding

Geometric series

In finance and economics many applications of geometric series exist. There are both finite and infinite series.

1. Summation for finite geometric series: a+ak+a,k-2.+…+a,k-n−1.=a,,k-n−1.-k−1.
2. Summation for infinite geometric series: $a + ak + ak^2 + \cdots + ak^{n-1} + \cdots = \frac{a}{1-k}$ or using other notation $\sum_{n=1}^{\infty} ak^{n-1} = \frac{a}{1-k}$, in both cases only when $|k| < 1$

An annuity is a sequence of equal payments made at fixed periods of time over some period. To calculate the present value of an annuity we use the formula for a finite geometric series:

$$P_n = \frac{a}{1+r} + \cdots + \frac{a}{(1+r)^n} = \frac{a}{r}\left[1 - \frac{1}{(1+r)^n}\right]$$

The future value of an annuity is the accumulated value in the account after n periods:
$F_n = \frac{a}{r}[(1+r)^n - 1]$
The two formulas for the present and future value are based on the idea of discrete accumulation. In the case of a continuous income stream, we have to use integrals to calculate the present and future values:

$$Present\ Discounted\ Value = \int_0^T f(t)e^{-rt}\,dt$$
$$Future\ Discounted\ Value = \int_0^T f(t)e^{-r(T-t)}\,dt$$
Discounted Value=,0-*T*-*f*,*t*.,*e*-−*r*(*t*−*s*)..*dt*, *over the interval* [*s*,*T*]

Repayment of Mortgage

Usually mortgages are paid back at a fixed interest rate, with equal payments due each period. The payments continue until the loan is paid off. The fixed payments go partly to the interest and partly to the outstanding principle. In the beginning the interest rate is large, but small in the last periods.

An example can be illustrative: A person has taken a mortgage for € A that he will pay over t installments at the rate of r% compounded annually. How much will he pay per installment? Let p be the amount for each installment and using the formula for a finite geometric serie:

$$\frac{p}{r} \times 100 \left[1 - \frac{1}{(1+\frac{r}{100})^n}\right] = A$$

Let us use the example of someone who has a mortgage of € 100000 that he will pay over 4 installments at the rate of 20% compounded annually. How much will he pay per installment?

$$\frac{p}{20} \times 100 \left[1 - \frac{1}{\left(1+\frac{20}{100}\right)^4}\right] = 100000$$

$p \times 5\left[1 - \frac{1}{(1.20)^4}\right] = 100000$, $p = \frac{100000}{5} \times 0.5177$, $p = 10354$

Now to find the amount per installment or p we can just use:

$$p = \frac{rA}{1-(1+r)^{-n}}$$

To find the number of periods required to pay back the loan at given amount per installment we can use:

$$n = \frac{lnp - \ln(p - rA)}{\ln(1+r)}$$

Internal Rate of Return

The internal rate of return is defined as an interest rate that makes the present value of all payments equal to zero. Just remember the following formula in which A is the initial investment and the returns per period are $p_1, p_2 \dots \dots \dots p_n$ for n periods, The rate of return is r. Then:

$$A = \frac{p_1}{(1+r)^1} + \frac{p_2}{(1+r)^2} + \cdots \dots \dots + \frac{p_n}{(1+r)^n}$$

To make the calculation easier assume that $(1+r)^{-1} = x$ and then rewrite the formula $A = p_1 x + p_2 x^2 + \cdots \dots + p_n x^n$. Then you can first solve for x and substitute $(1+r)^{-1} = x$ to find the value of r.

Difference equations

Difference equations are equations that relate certain quantities at different discrete moments of time. The initial period is t=0. A very simple difference equation is:

$x_{t+1} = a x_t$

Chapter 11 Many Variables

Two Variables

A function f of two real variables x and y with domain D is a rule that assigns a specified number $f(x,y)$ to each point (x,y) in D. In such a function there are two independent variables, and one dependent variable. The range still corresponds to the dependent variable.

The most widely used example in economics is the Cobb-Douglas function:
$F(x,y) = Ax^a y^b$

When differentiating a function with more than one variable, we first have to take so-called *partial derivatives* with respect to each independent variable. So if $z = f(x,y)$, then $\frac{\partial z}{\partial x}$ is the derivative of the function with respect to x, while y is kept constant. And $\frac{\partial z}{\partial y}$ is the derivative of the function with respect to y, while x is kept constant.

The formal definitions of the partial derivatives are:

1) $f_1'(x,y) = \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$, this partial derivative is approximately equal to the change in the original function $f(x,y)$ per unit increase in x, holding y constant.
2) $f_2'(x,y) = \lim_{k\to 0}\frac{f(x,y+k)-f(x,y)}{k}$, this partial derivative is approximately equal to the change in the original function $f(x,y)$ per unit increase in y, holding x constant.

The derivatives above are the first-order derivatives. We can also find derivatives of a higher order. The second-order derivatives are denotes as follows:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2},\ \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y},\ \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

We can see that on the second line the partial derivatives are actually the same. That means that it does not matter to which variable you differentiate first, if you differentiate for the other after. The order is not relevant.

If me move from a function with one independent variable to a function with two independent variables, then the geometric representation transforms from two-dimensional to three-dimensional. Instead of two axes, now there will be three. In the case of more variables we do not have the means to represent them as a graph visually.

Surfaces and Distance

In a three-dimensional figure a function represents a surface instead of a line. For a function with three variables (x, y, z), the general equation for a plane in space is given by $px + qy + rz = s$, in which $p, q, r \; and \; s$ are constants.

We can interpret each independent variable and each axis to represent a product, and the constants the corresponding prices. Then the surface is called budget plane, where p, q and r are cost of goods/unit and s is the total value. For a three-dimensional space, we can find the distance between two arbitrary points $(p_1, q_1, r_1) \; and \; (p_2, q_2, r_2)$ in that space. The distance is given by:

$$distance = \sqrt{(p_1, -p_2)^2 + (q_1 - q_2)^2 + (r_1 - r_2)^2}$$

The equation for a sphere is given for one center point (u, v, w) and using radius r:

$$(x-u)^2 + (y-v)^2 + (z-w)^2 = r^2$$

More Variables

To discuss functions with more than two independent variables we have to introduce the concept of a *vector*. Any ordered collection of numbers $(x_1, x_2, x_3, \dots, x_n)$ is called an n-vector. Vectors are usually denoted by bold letters. Hence, we can simply write $\boldsymbol{x} = (x_1, x_2, x_3, \dots, x_n)$. Formally, given any set D of n-vectors, a function of f of n variables $x_1, \dots, x_n$ with domain D is a rule that assigns a specifiek number $f(\boldsymbol{x}) = f(x_1, \dots, x_n)$ to each vector $\boldsymbol{x} = (x_1, x_2, x_3, \dots, x_n)$.

Functions with multiple variables can be linear, quadratic or exponential. Do not forget that an exponential function such as the Cobb-Douglas function can be transformed into a log-linear function by taking the natural logarithms:

$$F(x, y) = Ax^a y^b \rightarrow \ln F = \ln A + a \ln x + b \ln y$$

Again any function of n variables that can be constructed from continuous functions by combining addition, subtraction, multiplication, division and functional composition is continuous wherever it is defined.

Partial Derivatives

We can extend the differentiation process to functions with multiple variables. A way of representing the resulting sets of partial derivatives to the second-degree is in the *Hessian Matrix:*

$$f''(\boldsymbol{x}) = \begin{pmatrix} f''_{11}(\boldsymbol{x}) & \cdots & f''_{1n}(\boldsymbol{x}) \\ \vdots & \ddots & \vdots \\ f''_{n1}(\boldsymbol{x}) & \cdots & f''_{nn}(\boldsymbol{x}) \end{pmatrix}$$

Young's theorem says that if all the mth-order partial derivatives of a function are continuous, and if any two of them involve differentiating with respect to each of the variables the same number of times, then they are necessarily equal:

$$\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right) = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right)$$

Partial elasticities

Also in the case of functions with multiple variables we can find elasticities. If $z = f(x, y)$ then we define the partial elasticity of z with respect to x and y by:

$$El_x z = \frac{x}{z}\frac{\partial z}{\partial x} = \frac{\partial \ln z}{\partial \ln x}, and\ El_y z = \frac{y}{z}\frac{\partial z}{\partial y} = \frac{\partial \ln z}{\partial \ln y}$$

These expressions can be generalized to functions with more than two variables.

Chapter 12 Comparative Statistics

Chain Rule

When $z = F(x, y)$ with $x = f(t)$ and $y = g(t)$, then we can use the simple chain rule to find the *total derivative*:

$$\frac{dz}{dt} = F_1'(x,y)\frac{dx}{dt} + F_2'(x,y)\frac{dy}{dt}$$

The chain rule can also be applied to problems with multiple variables. If $z = F(x, y)$ and $x = f(t, s)$ and $y = g(t, s)$, then the composite function is z=F,f,t,s.,g,t,s.., then the two partial derivatives are :

$$\frac{dz}{dt} = F_1'(x,y)\frac{dx}{dt} + F_2'(x,y)\frac{dy}{dt}$$
$$\frac{dz}{ds} = F_1'(x,y)\frac{dx}{ds} + F_2'(x,y)\frac{dy}{ds}$$

We can transform these expressions into the general case with n variables:

$$\frac{dz}{dt_j} = \frac{\partial z}{\partial x_1}\frac{dx_1}{dt_j} + \frac{\partial z}{\partial x_2}\frac{dx_2}{dt_j} + \cdots + \frac{\partial z}{\partial x_n}\frac{dx_n}{dt_j}$$

Implicit Differentiation

Economists often have to differentiate functions that are defined implicitly by an equation, for example: F,x, y.=c and y is defined by an equation. To find the slope of the function $y = f(x)$ we can use:

$$y' = -\frac{F_1'(x,y)}{F_2'(x,y)} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$$

The process to find the second derivative of this derivative seems cumbersome, but is in essence the simple application of the quotient rule:

5) Write the expression of y' above as $y' = -\frac{G(x)}{H(x)}$.
6) Now differentiate to x: $y'' = -\frac{G'(x)H(x) - G(x)H'(x)}{[H(x)]^2}$
7) Both $G(x)$ and $H(x)$ are composite functions and therefore we can find the derivatives as follows:
8. $G'(x) = F_{11}''(x,y) \cdot 1 + F_{12}''(x,y) \cdot y'$

9. $H'(x) = F''_{21}(x,y) \cdot 1 + F''_{22}(x,y) \cdot y'$

10) Remember that $F''_{12} = F''_{21}$, and replace y' by $-\frac{F_1'}{F_2'}$, and insert all the results into y'' that we have found in step 2.

11) $y'' = -\frac{1}{(F_2')^3}[F''_{11}(F_2')^2 - 2F_1''F_1'F_2' + F''_{22}(F_1')^2]$ is the result we find

General Cases

In general, what we know about the partial derivatives when $z = f(x,y)$ and c is a constant is the following:

$$F(x,y,z) = c \;=>\; z'_x = -\frac{F'_x}{F'_z},\quad z'_y = -\frac{F'_y}{F'_z}\; for\; F'_z \neq 0$$

For the case with n variables we find:

$$F(x_1,\dots,x_n,z) = c \quad \Rightarrow \quad \frac{\partial z}{\partial x_i} = -\frac{\partial F/\partial x_i}{\partial F/\partial z}$$

Elasticity of Substitution

Even though in economics the slope of a curve is often downwards sloping, we change the sign of the slope and call it the *marginal rate of substitution of y for x (MRS):*

$$R_{xy} = \frac{F_x'(x,y)}{F_y'(x,y)} = -y' \approx -\frac{\Delta y}{\Delta x}$$

The elasticity of substitution σ_{yx} is the elasticity of the fraction $\frac{y}{x}$ with respect to the MRS. It is therefore more or less the percentage change in this fraction when we move along the level curve far enough for the MRS to change with 1%.

$$\sigma_{yx} = \mathrm{El}_{R_{xy}}\left(\frac{y}{x}\right)$$

Homogeneous Functions

A function $f(x,y)$ is said to be homogeneous of degree k if, for all (x,y) in D, the following holds for all $t > 0$:

$$f(tx,ty) = t^k f(x,y)$$

In words, multiplying both variables x and y by a factor t, will multiply the entire function by the factor t^k. One of the important properties of homogeneous functions is defined by *Euler's theorem*, which says that if a function is homogeneous of degree k, then:

$$xf_1'(x,y) + yf_2'(x,y) = kf(x,y)$$

In a more general formulation Euler's theorem looks like this:

$$\sum_{i=1}^{n} x_i f_i'(\boldsymbol{x}) = kf(\boldsymbol{x})$$

Given that f is a differentiable function in an open domain D.

A function is homothetic provided that:

$$\boldsymbol{xy} \in K,\ \ f(\boldsymbol{x}) = f(\boldsymbol{y}),\ \ t > 0 \ \ \Rightarrow \ \ f(t\boldsymbol{x}) = f(t\boldsymbol{y})$$

Every homogeneous function of degree k is also a homothetic function. More generally we can say that if H is a strictly increasing function and f is homogeneous of degree k, then *F*,***x***.=*H*,*f*,***x***…

Linear Approximation

The Linear approximation to $f(x,y)$ about (x_0,y_0) is given by the formula:

$$f(x,y) \approx f(x_o,y_o) + f'_1(x_o,y_o)(x - x_o) + f'_2(x_o,y_o)(y - y_o)$$

Where, $f'_1(x_o,y_o)$, stands for first derivative of (x_o,y_o) in terms of x and $f'_2(x_o,y_o)$ stands for the first derivative of (x_o,y_o), in terms of y.

We can generalize this equation for functions of several variables. For a function $z\ =\ f(\boldsymbol{x}) =\ f(x_1, \dots\dots\ x_n)\ \ about\ \ \boldsymbol{x}^0 = (x_1^0, \dots\dots, x_n^0)$, the linear approximation is:

$$f(\boldsymbol{x}) \approx f(\boldsymbol{x}^0) + \ f'_1(\boldsymbol{x}^0)(x - x^0) + \dots\dots\dots.. + f'_n(\boldsymbol{x}^0)(x_n - x_n^0)$$

In a situation with two independent variables we are no longer considering a two-dimensional graph, but a three-dimensional space. Therefore we can no longer speak about a tangent line, but have to consider a tangent plane. The tangent plane has the equation:

$$z - c = f'_1(a,b)(x - a) + \ f'_2(a,b)(y - b)$$

Differentials

Consider a function $z = f(x,y)$. When x and y change a bit, the resulting change in z is called the increment:

$$\Delta z = f(x + dx, y + dy) - f(x,y)$$

The differential of the function is denoted by dz or df and is:

dz=,*f*-1-′.,*x*,*y*.*dx*+,*f*-2-′.,*x*,*y*.*dy*

If dx and dy are small in absolute values then, $dz \approx \Delta z$. The following (familiar) rules apply to differentials (f and g are functions):

3. $d(af+bg)=a\,df+b\,dg$
4. $d(fg) = g\,df + f\,dg$
5. $d\left(\frac{f}{g}\right) = \frac{g\,df - f\,dg}{g^2}$
6. $z = g\big(f(x,y)\big) \Rightarrow dz = g'\big(f(x,y)\big)df$

Systems of Equations

In many economic models a large number of variables relate to each other through a system of simultaneous equations. To find the number of degrees of freedom we use the counting rule. First count the number of variables or n, and then how many number of 'independent' equations (i) there are. If $n > i$ then $n - i$ is the number of degrees of freedom. If $n < i$ then there is no solution to that system.

A system of equations with n number of variables will have k degrees of freedom. k is the number of variables that can be chosen freely. And $n - k$ is the number of variables whose value can be found when the value of k is decided upon.

In general, a system with as many equations as variables is usually consistent, in other words, has solutions, but it may have several solutions. Usually there is not a unique solution unless there are exactly as many equations as unknowns.

Basics of Differentiating Systems of Equation

To differentiate a system of equations, the first step to take is to differentiate both sides of the equation with respect to their variables. Consider for example the following two equations: $2a + 2b = 3x - 2y$. Step one would result in $2da + 2db = 3dx - 2dy$.
The next step is to find the values of $da \text{ and } db$ in terms of $dx \text{ and } dy$ by using the two equations. Then we have to find a'_x and b'_x also a'_y and $b'_{y'}$. Just remember that if the equation for da looks like $da = 3dx + 4dy$ then the result would be simply $a'_x = 3$ and $a'_y = 4$.
The last thing to do is to substitute the points that are given to find the exact values

Chapter 13 Optimization

Two Variables

We can easily extend the problem of optimization of a function with one variable to optimization of a function with two variables. A differential function $z = f(x, y)$ can have a maximum or a minimum at an interior point (x_0, y_0) in a set S only if it is a stationary point. In other words, it is a necessary condition for an interior extreme point to satisfy the *first order conditions (FOCs)* to be an optimum:

$$f_1'(x, y) = 0 \quad and \quad f_2'(x, y) = 0$$

There are also conditions for a potential maximum or a minimum that are sufficient, but not necessary:

17. If for all (x, y) in S the following inequalities hold, then function is concave and the point (x_0, y_0) is a maximum point of the function:
18. $f_{11}''(x, y) \leq 0$
19. $f_{22}''(x, y) \leq 0$
20. $f_{11}''(x, y) f_{22}''(x, y) - \left(f_{12}''(x, y)\right)^2 \geq 0$
21. If for all (x, y) in S the following inequalities hold, then the function is convex and the point (x_0, y_0) is a minimum point of the function:
22. $f_{11}''(x, y) \geq 0$
23. $f_{22}''(x, y) \geq 0$
24. $f_{11}''(x, y) f_{22}''(x, y) - \left(f_{12}''(x, y)\right)^2 \geq 0$

Local Optima

At any local extreme point in the interior of a domain of a differentiable function, the function must be stationary, which means that all its first-order conditions need to be equal to zero. A point is said to be a local maximum point when $f(x, y) \leq f(x_0, y_0)$ for all pairs of x and y that lie sufficiently close to that particular point (x_0, y_0). For a local minimum the reasoning is the other way around. Note that a global maximum or a global minimum are also a local maximum or local minimum, but the other way around is not true.

A so-called *saddle point* is a stationary point with the property that there exist points (x, y)

arbitrarily close to (x_0,y_0) with $f(x,y) < f(x_0,y_0)$, but also points with $f(x,y) > f(x_0,y_0)$. Therefore a stationary point, when the FOCs are equal to zero, can be a local maximum, a local minimum and a saddle point. To determine the nature of such a stationary point we need the *second-order conditions*. Again, these conditions are sufficient but not necessary.

- (x_0,y_0) is a strict local maximum if

    2. $f''_{11}(x_0,y_0) < 0$

    3. $f''_{11}(x_0,y_0)f''_{22}(x_0,y_0) - \left(f''_{12}(x_0,y_0)\right)^2 > 0$

- (x_0,y_0) is a local minimum if

    5. $f''_{11}(x_0,y_0) > 0$

    6. $f''_{11}(x_0,y_0)f''_{22}(x_0,y_0) - \left(f''_{12}(x_0,y_0)\right)^2 > 0$

- (x_0,y_0) is a saddle point if

    8. $f''_{11}(x_0,y_0)f''_{22}(x_0,y_0) - \left(f''_{12}(x_0,y_0)\right)^2 < 0$

- (x_0,y_0) could be any of the three options if

    10. $f''_{11}(x_0,y_0)f''_{22}(x_0,y_0) - \left(f''_{12}(x_0,y_0)\right)^2 = 0$

The Extreme Value Theorem

Consider a function $f(x,y)$ that is continuous throughout a nonempty, closed and bounded set. Then there exist both a point (a,b) in the set where the function has a minimum and a point (c,d) in the set where the function has a maximum. In a more formal way:

$$f(a,b) \leq f(x,y) \leq f(c,d)$$

A point (a,b) is called an interior point of a set in the plane if there exists a circle centered at this point such that all points inside this circle are still part of the set. This circle can be extremely small. An *open* set consists only of such interior points. A set is called a *closed* set when it contains all its *boundary points*. A point is a boundary point if every circle (every size) contains points of the set as well as points of its complement. In other words, a set is closed if and only if its complement is open. A set can also be neither open nor closed, when a set contains some boundary points but not all. Illustrations you can find on page 483.

A set is *bounded* if the whole set is contained within a sufficiently large circle. Otherwise a set is unbounded. A set that is both closed and bounded is called a *compact* set.

The following steps need to be taken to find the maxima and minima of a differentiable function defined on a compact set:

1. First find the stationary points of the function $f(x,y)$ by finding the FOCs.
2. Then find the largest and smallest value of the function on the boundary of the set.
3. Compute the values of the function at all the points found in 1 and 2. The largest value is the maximum and the smallest value is the minimum.

Three or More Variables

The optimization problem can be generalized to problems of three or more variables. The first order conditions are formulated as $f_i'(\mathbf{x}) = 0$. The extreme value theorem is also valid if the function is continuous throughout a nonempty and compact set: $f(\mathbf{d}) \leq f(\mathbf{x}) \leq f(\mathbf{c})$ for all **x** in the set.

An important and useful result is that maximizing a function is equivalent to maximizing a strictly increasing transformation of that same function. Formally speaking, if $g(\mathbf{x}) = F\big(f(\mathbf{x})\big)$, then

1. If F is increasing and $\boldsymbol{c}$ maximizes (or minimizes) the function f over the set, then $\boldsymbol{c}$ also maximizes (or minimizes) g over the set.
2. If F is strictly increasing, then $\boldsymbol{c}$ maximizes (or minimizes) f over the set if and only if $\boldsymbol{c}$ maximizes (or minimizes) g over the set.

For example, differentiating $f(x,y) = e^{x^2+2xy^2-y^3}$ has exactly the same solutions as differentiating $g(x,y) = x^2 + 2xy^2 - y^3$.

Envelope theorem

In economics many problems depend on changing parameters that are held constant in the process of optimization, such as prices, tax rates, income levels etc. We may want to know however how optimal values change when these parameters change.

Consider the optimization problem of a function $\max_x f(x,r)$ that we can rewrite as $f^*(r) = f(x^*(r),r)$ because the optimum value of x, x*, depends on parameter r. This last function is called the *value function*. Differentiation to r gives us the following equation:

$$\frac{df^*(r)}{dr} = f_1'(x^*(r),r)\frac{dx^*(r)}{dr} + f_2'(x^*(r),r)$$

$$\frac{df^*(r)}{dr} = f_2'(x^*(r), r)$$

The generalization of this result is called the envelope theorem. If $f^*(\boldsymbol{r}) = \max_{\boldsymbol{x}} f(\boldsymbol{x}, \boldsymbol{r})$ and if $\boldsymbol{x}^*(\boldsymbol{r})$ is the value of **x** that maximizes the function $f(\boldsymbol{x}, \boldsymbol{r})$, then:

$$\frac{\partial f^*(\boldsymbol{r})}{\partial r_j} = \left[\frac{\partial f(\boldsymbol{x}, \boldsymbol{r})}{\partial r_j}\right]_{\boldsymbol{x}=\boldsymbol{x}^*(\boldsymbol{r})}$$

Chapter 14 Constrained Optimization

Lagrangian Multiplier Method

A maximization problem can be constrained. An example from economics is the maximization of a utility function under the constraint of a budget. We would write that problem as $\max u(x,y)$ subject to $px + y = m$. Economists make use of the Lagrange multiplier method to solve such complicated constrained maximization problems:

To find the solutions of the problem maximize, or minimize, $f(x,y)$ subject to $g(x,y) = c$ proceed as follows:

4. Write down the Lagrangian: $\mathcal{L}(x,y) = f(x,y) - \lambda(g(x,y) - c)$ where λ is a constant.

5. Differentiate this Lagrangian $\mathcal{L}$ with respect to x and y and equate both partial derivatives to zero.

6. These two equations together with the constraint give this system of equations (FOCs):

    7. $\mathcal{L}_1'(x,y) = f_1'(x,y) - \lambda g_1'(x,y) = 0$

    8. $\mathcal{L}_2'(x,y) = f_2'(x,y) - \lambda g_2'(x,y) = 0$

    9. $g(x,y) = 0$

10. Solve these three equations simultaneously for the three unknowns x, y and λ. These triplets (x, y, λ) are solution candidates.

The Lagrange multiplier λ is the rate at which the optimal value of the objective function changes with respect to changes in the constraint constant. Economists call λ the shadow price.

This method is based on Lagrange's theorem. For completeness, we formally describe the theorem:

Suppose that $f(x,y)$ and $g(x,y)$ have continuous partial derivatives in a domain A of the xy-plane, and that the point (x_0,y_0) is both an interior point of A and a local extreme point for $f(x,y)$ subject to the constraint $g(x,y) = c$. Suppose further that $g_1'(x_0,y_0)$ and $g_2'(x_0,y_0)$ are not both 0. Then there exists a unique number λ such that the Lagrangian $\mathcal{L}(x,y) = f(x,y) - \lambda(g(x,y) - c)$ has a stationary point at (x_0,y_0).

Sufficient conditions

The process above provides us with the necessary conditions for the solution. In order to confirm the found candidates as solutions, we need to carefully check the nature of the points.

We know that if the Lagrangian is concave, then the point (x_0, y_0) is a maximum. If the Lagrangian is convex, then the point (x_0, y_0) is a minimum. Formally, we need to make the following calculation. Point (x_0, y_0) is a:

4. local max if $(f''_{11} - \lambda g''_{11})(g'_2)^2 - 2(f''_{12} - \lambda g''_{12})g'_1 g'_2 + (f''_{22} - \lambda g''_{22})(g'_1)^2 < 0$

5. local min if $(f''_{11} - \lambda g''_{11})(g'_2)^2 - 2(f''_{12} - \lambda g''_{12})g'_1 g'_2 + (f''_{22} - \lambda g''_{22})(g'_1)^2 > 0$

It is possible and straightforward to complicate the Lagrangian by adding constraints or variables.

Nonlinear Programming

Problems of nonlinear programming concern inequalities. A simply inequality constraint is when certain variables cannot be negative, for example $x_i \geq 0$. The nonlinear programming problem is denoted as:

$$\max f(x, y) \text{ subject to } g(x, y) \leq c$$

The Lagrangian method does not change, but the candidates that are to be found, should satisfy the inequality constraint.

The problem can also involve a set of inequality constraints:

$$\max f(x_i, \dots, x_n) \text{ subject to } \begin{cases} g_1(x_i, \dots, x_n) \leq c_1 \\ \dots \\ g_m(x_i, \dots, x_n) \leq c_m \end{cases}$$

The set of vectors $\boldsymbol{x} = (x_i, \dots, x_n)$ that satisfy all the constraints is called the *admissible set* or the *feasible set*.

Chapter 15 Matrices and Vectors

Matrices

A system of equations is consistent if it has at least one solution. When the system has no solutions at all, then it is called inconsistent.

A matrix is a rectangular array of numbers considered as a mathematical object. Matrices are often used to solve systems of equations. When the matrix consists of m rows and n columns then the matrix is said to have the order $m \times n$. All the numbers in a matrix are called elements or entries. If $m = n$ then the matrix is called a square matrix. The main diagonal runs from the top left to the bottom right and are the elements $a_{11}, a_{22}, a_{33} \dots$

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{m1} & \dots & a_{mn} \end{pmatrix}$$

A matrix with only one row, or only one column, is called a *vector*. And we can distinguish between a row vector and a column vector. A vector is usually denoted by a bold letter $\boldsymbol{x}$.

One can transform a system of equations into a matrix or order the coefficients of system in a matrix.

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$
$$\dots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

This system can be written, now in short form, as: $\boldsymbol{Ax} = \boldsymbol{b}$.

Matrix Operations

Two matrices $\boldsymbol{A} = (a_{ij})_{m\times n}$ and $\boldsymbol{B} = (b_{ij})_{m\times n}$ are said to be equal if all $a_{ij} = b_{ij}$, or in words, if they have the same order and if all corresponding entries are equal. Otherwise they are not equal and we write $\boldsymbol{A} \neq \boldsymbol{B}$.

The reasoning for addition and multiplication by a constant is straightforward.

$$\boldsymbol{A} + \boldsymbol{B} = (a_{ij})_{m\times n} + (b_{ij})_{m\times n} = (a_{ij} + b_{ij})_{m\times n}$$

$$\alpha\boldsymbol{A} = \alpha(a_{ij})_{m\times n} = (\alpha a_{ij})_{m\times n}$$

The rules that are related to these two operations are:

3. $(\boldsymbol{A}+\boldsymbol{B})+\boldsymbol{C}=\boldsymbol{A}+(\boldsymbol{B}+\boldsymbol{C})$
4. $\boldsymbol{A}+\boldsymbol{B}=\boldsymbol{B}+\boldsymbol{A}$
5. $\boldsymbol{A}+\boldsymbol{0}=\boldsymbol{A}$
6. $\boldsymbol{A}+(-\boldsymbol{A})=\boldsymbol{0}$
7. $(\alpha+\beta)\boldsymbol{A}=\alpha\boldsymbol{A}+\beta\boldsymbol{B}$
8. $\alpha(\boldsymbol{A}+\boldsymbol{B})=\alpha\boldsymbol{A}+\alpha\boldsymbol{B}$

Matrix Multiplication

For multiplication of two matrices, suppose that $\boldsymbol{A}=(a_{ij})_{m\times n}$ and $\boldsymbol{B}=(b_{ij})_{m\times n}$. Then the product $\boldsymbol{C}=\boldsymbol{AB}$ is the $m\times p$ matrix $\boldsymbol{C}=(c_{ij})_{m\times p}$. The element in the i'th row and j'th column is the product of:

$$c_{ij}=\sum_{r=1}^{n}a_{ir}b_{rj}=a_{i1}b_{1j}+a_{i2}b_{2j}+\cdots+a_{ik}b_{kj}+\cdots+a_{in}b_{nj}$$

To help visualizing this summation, have a look at the following multiplication of matrices:

$$\boldsymbol{A}=\begin{pmatrix}0&1&2\\2&3&1\\4&-1&6\end{pmatrix},\boldsymbol{B}=\begin{pmatrix}3&2\\1&0\\-1&1\end{pmatrix}$$

$$\boldsymbol{AB}=\begin{pmatrix}0&1&2\\2&3&1\\4&-1&6\end{pmatrix}\begin{pmatrix}3&2\\1&0\\-1&1\end{pmatrix}=\begin{pmatrix}-1&2\\8&5\\5&14\end{pmatrix}$$

In this case $\boldsymbol{AB}$ is defined but $\boldsymbol{BA}$ would not be defined, because in that case the number of elements in the rows of B does not match the number of elements in the columns of A. Even if they are both defined, they are not automatically equal.

There are rules for matrix multiplication:

1. $(\boldsymbol{AB})\boldsymbol{C}=\boldsymbol{A}(\boldsymbol{BC})$, Associative law
2. $\boldsymbol{A}(\boldsymbol{B}+\boldsymbol{C})=\boldsymbol{AB}+\boldsymbol{AC}$, Left distributive law
3. $(\boldsymbol{A}+\boldsymbol{B})\boldsymbol{C}=\boldsymbol{AC}+\boldsymbol{BC}$, Right distributive law
4. $(\alpha\boldsymbol{A})\boldsymbol{B}=\boldsymbol{A}(\alpha\boldsymbol{B})=\alpha(\boldsymbol{AB})$
5. $\boldsymbol{A}^n=\boldsymbol{AA}\ldots\boldsymbol{A}$, A is repeated n times

There are some dangerous mistakes that are often maid:

1. $\boldsymbol{AB} \neq \boldsymbol{BA}$
2. $\boldsymbol{AB} = \boldsymbol{0}$ Does not imply that either $\boldsymbol{A}$ or $\boldsymbol{B}$ is $\boldsymbol{0}$
3. $\boldsymbol{AB} = \boldsymbol{AC}$ And $\boldsymbol{A} \neq \boldsymbol{0}$ do not imply that $\boldsymbol{B} = \boldsymbol{C}$

The Identity Matrix and the Transpose

The identity matrix of order n, denoted by $\boldsymbol{I}_n$, is the matrix having only ones along the main diagonal and zero's elsewhere:

$$\boldsymbol{I}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The identity matrix is in practice the equivalent of 1 in the numerical system, because $\boldsymbol{AI}_n = \boldsymbol{I}_n\boldsymbol{A} = \boldsymbol{A}$.

The transpose of matrix $\boldsymbol{A}$, $\boldsymbol{A}'$, is the mirror matrix of $\boldsymbol{A}$. More formally, $\boldsymbol{A}'$ is defined as the $n \times m$ matrix whose first column is the first row of $\boldsymbol{A}$, whose second column is the second row of $\boldsymbol{A}$, and so on. Thus:

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \Rightarrow \boldsymbol{A}' = \begin{pmatrix} a_{11} & \cdots & a_{m1} \\ \vdots & \ddots & \vdots \\ a_{1n} & \cdots & a_{mn} \end{pmatrix}$$

The rules for transposition are:

3) $(\boldsymbol{A}')' = \boldsymbol{A}$
4) $(\boldsymbol{A} + \boldsymbol{B})' = \boldsymbol{A}' + \boldsymbol{B}'$
5) $(\alpha\boldsymbol{A})' = \alpha\boldsymbol{A}'$
6) $(\boldsymbol{AB})' = \boldsymbol{B}'\boldsymbol{A}'$

A matrix is called symmetric when $\boldsymbol{A} = \boldsymbol{A}'$.

Gaussian Elimination

One method of solving systems of equations is by elimination of the unknowns. Elementary row operations can transform equations in such a way that unknowns can be eliminated. There are three kinds of elementary row operations:

4. Interchange any pair of rows

5. Multiply any row by a scalar
6. Add any multiple of one row to a different row

Strictly the Gaussian method of elimination involves three steps:

4. Make a staircase with 1 as the coefficient for each nonzero leading entry.
5. Produce 0's above each leading entry.
6. Express the unknowns in terms of those unknowns that do not occur as leading entries. The number of unknowns that can be chosen freely is the number of *degrees of freedom*.

See page 565 to 569 for extensive numerical examples.

Vectors

The numbers in a vector are called the components, or coordinates of the vector. A vector is just a specific type of matrix and therefore, the algebra of matrices is also valid for vectors:

1. Two vectors are equal only if all their corresponding components are equal.
2. The sum of two n-vectors is found by adding each component of the first vector to the corresponding component in the other vector.
3. Any vector can be multiplied by a real number.
4. The difference between two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is defined as $\boldsymbol{a} - \boldsymbol{b} = \boldsymbol{a} + (-1)\boldsymbol{b}$.

The so-called *inner product* of the n-vectors $\boldsymbol{a} = (a_1, a_2, \dots, a_n)$ and $\boldsymbol{b} = (b_1, b_2, \dots, b_n)$ is defined as:

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n = \sum_{i=1}^{n} a_i b_i$$

The rules for the inner product are, if $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are n-vectors and α is a scalar then:

1. $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$
2. $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$
3. $(\alpha \boldsymbol{a}) \cdot \boldsymbol{b} = \boldsymbol{a} \cdot (\alpha \boldsymbol{b}) = \alpha(\boldsymbol{a} \cdot \boldsymbol{b})$
4. $\boldsymbol{a} \cdot \boldsymbol{a} > 0 \iff \boldsymbol{a} \neq \boldsymbol{0}$

For a small section on geometric interpretations of vectors, have a look at page 575.

Chapter 16 Determinants and Inverse Matrices

Determinants

The value of the determinant of a matrix $\boldsymbol{A}$ denoted by $\det(\boldsymbol{A})$ or $|\boldsymbol{A}|$ determines if there is a unique solution to the corresponding system of equations.

$$|\boldsymbol{A}| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

This particular example is said to have the order two. Calculating the determinant of order two is simple. When a determinant is of a higher order though, the calculations become more extensive. Take for example this determinant of order 3:

$$|\boldsymbol{A}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

This expression is actually the same expression as:

$$|\boldsymbol{A}| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

A numerical example illustrates the procedure. Calculate:

$$|\boldsymbol{A}| = \begin{vmatrix} 3 & 0 & 2 \\ -1 & 1 & 0 \\ 5 & 2 & 3 \end{vmatrix}$$

The solution is:

$$|\boldsymbol{A}| = 3\begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} - 0\begin{vmatrix} -1 & 0 \\ 5 & 3 \end{vmatrix} + 2\begin{vmatrix} -1 & 1 \\ 5 & 2 \end{vmatrix} = 3 \cdot 3 - 0 + 2 \cdot (-2 - 5) = -5$$

To determine the sign of any term in the sum above, mark in the array all the elements appearing in that term. Join all possible pairs of these elements with lines. These lines will then either fall or rise to the right. If the number of rising lines is even, then the corresponding term is assigned a plus sign, if it is odd a minus sign. (Illustration on page 593)

In fact, the definition of the determinant is a sum of $n!$ terms where each term is the product of n elements of the matrix, with one element from each row and one element from each column. Moreover, every product of n factors in which each row and each column is represented exactly once, must appear in the sum.

Basic Rules

Consider the $n \times n$ matrix $\boldsymbol{A}$, then:

- If all the elements in a row or column of $\boldsymbol{A}$ are 0, then $|\boldsymbol{A}| = 0$.
- $|\boldsymbol{A}'| = |\boldsymbol{A}|$ where $\boldsymbol{A}'$ is the transpose of $\boldsymbol{A}$.
- If all the elements in a single row or column of $\boldsymbol{A}$ are multiplied by a certain number, then the determinant is also multiplied by this same number.
- If two rows or two columns of $\boldsymbol{A}$ are interchanged, the determinant changes sign, but the absolute value remains unchanged.
- If two of the rows or columns of $\boldsymbol{A}$ are proportional, then $|\boldsymbol{A}| = \boldsymbol{0}$.
- The value of the determinant of $\boldsymbol{A}$ is unchanged if a multiple of one row or one column is added to a different row or column of $\boldsymbol{A}$.
- The determinant of the product of two $n \times n$ matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ is the product of the determinants of each of the factors: $|\boldsymbol{AB}| = |\boldsymbol{A}| \cdot |\boldsymbol{B}|$
- If α is a real number, then $|\alpha\boldsymbol{A}| = \alpha^n|\boldsymbol{A}|$

Note that in general $|\boldsymbol{A} + \boldsymbol{B}| \neq |\boldsymbol{A}| + |\boldsymbol{B}|$.

Expansion by Cofactors

Each term of the determinant of matrices contains one element from each row and one element from each column. The expansion of $|\boldsymbol{A}|$ in terms of elements of the row i is called the *cofactor expansion*:

$$|\boldsymbol{A}| = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{ij}C_{ij} + \cdots + a_{in}C_{in}$$

the cofactor expansion can also be found for a certain column. To find each cofactor C_{ij} there is a simple procedure to apply to the determinant. First delete row i and column j to arriver at a determinant of order $n-1$, called a *minor*. Then, multiply the minor by the factor $(-1)^{i+j}$:

$$C_{ij} = (-1)^{i+j} \begin{vmatrix} a_{11} & \cdots & a_{1,j-1} & \cancel{a_{1j}} & a_{1,j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2,j-1} & \cancel{a_{2j}} & a_{2,j+1} & \cdots & a_{2n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \cancel{a_{i1}} & & \cancel{a_{i,j-1}} & \cancel{a_{ij}} & \cancel{a_{i,j+1}} & & \cancel{a_{in}} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & \cancel{a_{nj}} & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix}$$

Inverse

Given a matrix $\boldsymbol{A}$ we say that $\boldsymbol{X}$ is an inverse of $\boldsymbol{A}$, and vice versa, if there exists a matrix $\boldsymbol{X}$ such that:

$$\boldsymbol{AX} = \boldsymbol{XA} = \boldsymbol{I}$$

$\boldsymbol{I}$ is the identity matrix. In this case the matrix $\boldsymbol{A}$ is said to be *invertible*. But note that only square matrices have inverses. A square matrix is said to be singular if its determinant equals zero and nonsingular if its determinant does not equal zero.

A matrix has an inverse if and only if it is nonsingular:

$$\boldsymbol{A} \; has \; an \; inverse \iff |\boldsymbol{A}| \neq 0$$

If a matrix has an inverse, then it is unique. So, assuming that the determinant is nonsingular, the following result has been proven:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \implies \boldsymbol{A}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

The inverse has four properties. Let A and B be invertible $n \times n$ matrices, then:

- ⇨ $\boldsymbol{A}^{-1}$ is invertible and $(\boldsymbol{A}^{-1})^{-1} = \boldsymbol{A}$
- ⇨ $\boldsymbol{AB}$ is invertible and $(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$
- ⇨ The transpose $\boldsymbol{A}'$ is invertible and $(\boldsymbol{A}')^{-1} = (\boldsymbol{A}^{-1})'$
- ⇨ $(c\boldsymbol{A})^{-1} = c^{-1}\boldsymbol{A}^{-1}$ whenever c is not zero

Provided that $|\boldsymbol{A}| \neq 0$, the following holds:

$$\boldsymbol{AX} = \boldsymbol{B} \iff \boldsymbol{X} = \boldsymbol{A}^{-1}\boldsymbol{B}$$
$$\boldsymbol{YA} = \boldsymbol{B} \iff \boldsymbol{Y} = \boldsymbol{BA}^{-1}$$

Cramer's Rule

Cramer's rule is that a general linear system of equations with n equations and n unknowns has a unique solution if and only if $\boldsymbol{A}$ is nonsingular, hence if the determinant does not equal zero.

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\ldots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

The solution is:

$$x_1 = \frac{D_1}{|\boldsymbol{A}|}, x_2 = \frac{D_2}{|\boldsymbol{A}|}, \ldots, x_n = \frac{D_n}{|\boldsymbol{A}|}$$

Where D denotes the determinant obtained from the determinant of A by replacing a column by the b_n's and then calculating its value through cofactor expansion.

If all the b equal zero, then the system of equations is called homogeneous. A homogeneous system will always have the so-called *trivial solution* which is $x_1 = x_2 = \cdots = x_n = 0$. Often one is interested in finding nontrivial solutions of a homogeneous system.

Chapter 17 Linear Programming

General

The general linear programming problem is that of maximizing or minimizing the objective function:

$$z = c_1 x_1 + \cdots + c_n x_n$$

This objective function is subject to a set of inequality constraints:

$$a_{11}x_1 + \cdots + a_{1n}x_n \leq b_1$$
$$a_{21}x_1 + \cdots + a_{2n}x_n \leq b_2$$
$$\dots \dots \dots$$
$$a_{m1}x_1 + \cdots + a_{mn}x_n \leq b_m$$

Usually it is explicitly assumed that the variables cannot be negative, the nonnegativity constraint. The vector of n solutions, that satisfies these constraints, is called the feasible or admissible vector.

Duality Theory

When an economist is confronted with an optimization problem there is are two ways of approaching the problem. A maximization problem has a mirror that is a minimization problem. For example, if the problem involves the allocation of scarce resources he can try to maximize the production constraint by the available scarce resources, or he can try to minimize the use of resources given a level of production. Hence, there is a duality involved.

In general, consider the general linear programming problem, also called the *primal* problem:

$$\max c_1 x_1 + \cdots + c_n x_n \ \ s.t. \ \begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n \leq b_1 \\ \dots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \leq b_m \end{cases}$$

Then the *dual* problem is:

$$\min b_1 u_1 + \cdots + b_n u_n \ \ s.t. \ \begin{cases} a_{11}u_1 + \cdots + a_{1n}u_n \leq c_1 \\ \dots \\ a_{m1}u_1 + \cdots + a_{mn}u_n \leq c_m \end{cases}$$

For both problems the nonnegativity constraint holds as well.

Suppose the primal problem has an optimal solution, then the dual problem also has an optimal solution and the corresponding values of the objective functions are equal. If the prima has no bounded optimum, then the dual has no feasible solution. Symmetrically, if the primal problem has no feasible solution, then the dual has no bounded optimum.